

# Emergence and non-typicality of the finiteness of the attractors in many topologies

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## Abstract

We will introduce the notion of Emergence for a dynamical system, and we will conjecture the local typicality of super-polynomial ones. Then, as part of this program, we will provide sufficient conditions for an open set of  $C^d$ -families of  $C^r$ -dynamics to contain a Baire generic set formed by families displaying infinitely many sinks at every parameter, for all  $\infty \geq r \geq d \geq 1$  and  $d < \infty$  and two different topologies on families. In particular the case  $d = r = 1$  is new.

## 1 Introduction

### 1.1 Tentatives to describe typical dynamics

Under the dual leadership of Anosov-Sinai in USSR and Smale in the USA, the hyperbolic theory for differentiable dynamical systems grew up. We shall recall some elements of this theory.

Let  $M$  be a manifold and let  $f$  be a  $C^1$ -diffeomorphisms of  $M$ . A compact set  $K \subset M$  is hyperbolic if the tangent space of  $TM|K$  is split into two vector sub-bundles  $E^s$  and  $E^u$ , which are both  $Df$ -invariant and respectively contracted and expanded by the dynamics:

$$TM|K = E^s \oplus E^u \quad Df(E^s) = E^s \quad Df(E^u) = E^u \quad \lim_{+\infty} \|Df^n|E^s\| = 0 \quad \lim_{+\infty} \|Df^{-n}|E^u\| = 0$$

There are many examples of hyperbolic sets such as the Anosov maps (when  $K = M$ ), the Smale Horseshoes, the Derivated of Anosov, and the Plykin attractors for diffeomorphisms, see [Sma67] for more details.

An important property of the hyperbolic sets is their structural stability:

**Theorem 1** (Anosov [Ano67]). *For every  $C^1$ -perturbation  $f'$  of  $f$ , there is a unique hyperbolic set  $K'$  for  $f'$ , which is homeomorphic to  $K$  via a map  $h: K \rightarrow K'$   $C^0$ -close to the canonical inclusion  $K \hookrightarrow M$  and which conjugates the dynamics:*

$$h \circ f|K = f'|K' \circ h|K.$$

The hyperbolic set  $K$  is a *basic set* if it is *transitive* (there is a dense orbit in  $K$ ) and *locally maximal*: there is a neighborhood  $N$  of  $K$  such that  $K = \bigcap_{n \in \mathbb{Z}} f^n(N)$ .

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Smale defined the diffeomorphisms satisfying *Axiom A* as those whose non-wandering set<sup>1</sup> is the finite disjoint union of basic sets.

A basic set  $K$  is a *hyperbolic attractor* if  $K = \bigcap_{n \geq 0} f^n(N)$ . Another important result of this theory is:

**Theorem 2** (Sinai-Bowen-Ruelle). *Given a hyperbolic attractor, there exists a unique invariant probability measure  $\nu$  on  $N$  so that for Lebesgue almost every  $z \in N$ , for every continuous function  $\phi$ :*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(z)) = \int \phi d\mu .$$

This result was very appreciated by physicists since it enables one to get from a deterministic system a new system with robust statistical properties, and somehow to make a conceptual bridge between Classical Mechanics and Statistical mechanics.

Also Smale made the following conjecture:

**Conjecture 3** (Smale 1965). *A Baire generic<sup>2</sup> diffeomorphism of a compact manifold satisfies Axiom A.*

This conjecture (and others by Smale and by Thom) appear at the beginning of a mathematical optimistic movement aiming to describe a typical dynamical system.

However, Smale and Smale-Abraham (1966) found soon a counter example to this conjecture.

In 1974, a student of Smale, Newhouse discovered an extremely complicated new phenomenon, occurring in a locally Baire generic set of dynamics.

**Theorem 4** ([New74]). *For every  $r \geq 2$ , for every manifold  $M$  of dimension  $\geq 2$ , there exist a non-empty open set  $U \subset \text{Diff}^r(M)$  and a generic set  $\mathcal{R} \subset U$  so that for every  $f \in \mathcal{R}$ , the dynamics  $f$  has infinitely many sinks, each of which having very different statistical properties.*

Clearly these dynamics do not satisfy Axiom A (which have finitely many attractors). Even today, we do not know to describe a single example of these dynamics – in the meaning that – we do not know even if Lebesgue almost every point belongs to the basin of an *ergodic attractor*.

From [PS89], an *ergodic attractor*  $(\Lambda, \mu)$  is a compact transitive set  $\Lambda$  supporting an invariant probability measure  $\mu$  s.t. for a set of positive Lebesgue measure  $B$  (called *Basin*) it holds:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(z)) = \int \phi d\mu, \quad \forall \phi \in C^0(M, \mathbb{R}) \quad \forall z \in B.$$

In the mean time the simulations of the atmospheric physicist Lorenz showed a new chaotic attractor for an ODE [Lor63]. Later Hénon modeled the first return of map of this flow, to get a simple paradigmatic example of a chaotic surface map: the Hénon map  $(x, y) \mapsto (x^2 + a + y, -bx)$  parametrized by  $a, b \in \mathbb{R}$ . He conjectured that for  $b = 0.3$  and a certain parameter  $a$  this map has a chaotic attractor [Hén76]. During a series of papers, this conjecture have been shown to be true for  $b$  sufficiently small.

**Theorem 5** (Benedicks-Carleson [BC91], Mora-Viana [MV93], Benedicks-Young [BY93], Wang-Young [YW01], Takahasi [Tak11], Berger [Bera], Yoccoz (1990-Today)). *For  $b$  sufficiently small, for a set of Lebesgue positive measure of parameters  $a$ , the map  $(x, y) \mapsto (x^2 + a + y, -bx)$  has unique ergodic attractor  $(\Lambda, \mu)$ , which is not supported by an attracting periodic orbit.*

<sup>1</sup>The set of point  $z \in M$  so that any neighborhood  $V$  of  $z$  intersects one of its iterates:  $\exists n \neq 0 : f^n(V) \cap V \neq \emptyset$ .

<sup>2</sup>A set is Baire generic if it is equal to a countable intersection of open dense sets.

The conjecture of Hénon was a posteriori disturbing since an arbitrarily small neighborhood  $N$  of the attractor is a topological disk, and so the attractor cannot be a hyperbolic attractor (otherwise there would be a line field on the topological disk  $N$ ). Also the simplicity of the model, its physical meaning and the concept of abundance involved make this phenomenon unavoidable.

That is why the next conjectures have been formulated thanks to the concept of typicality sketched by Kolmogorov during his plenary talk in the ICM 1954. Here is a version of typicality which appears in many conjectures:

**Definition 6** (Arnold-Kolmogorov typicality). *A property  $\mathcal{P}$  on dynamics of a manifold  $M$  is typical if there exists a Baire generic set of  $C^d$ -families  $(f_a)_{a \in \mathbb{R}^k}$  of  $C^r$ -dynamics so that  $\mathcal{P}$  is satisfied by Lebesgue almost every small parameter  $a$ .*

Hence this definition of typicality involved integers  $k, d, r$ . We will discuss about the topological spaces of families in the next section.

To take into account the aforementioned examples and counter examples, there were several conjectures claiming the typicality of the finiteness of attractors, let us recall the following<sup>3</sup>:

**Conjecture 7** (Pugh-Shub [PS96]). *Typically (in the sens of Arnold-Kolmogorov) a diffeomorphism of a compact manifold has a finite numbers of topological attractors (and so sinks).*

These conjectures aimed to model typical dynamics thanks to finitely many attractors. The general strategy though to prove them was to study the unfolding of stable and unstable manifolds (in analogy with Thom-Mather works in singularity Theory).

Recently, in [Ber16b], a mechanism has been found to stop the unfolding for an open set of dynamics' families. This mechanism is given by the *parablender*, a generalisation of Bonatti-Diaz Blender for parameter families. This enabled to prove:

**Theorem 8** ([Ber16b, Ber16a]). *For every manifold of dimension at least 3, for every  $k \geq 0$ , for every  $r > d \geq 1$ , there exists an open set of  $\hat{\mathcal{U}}$  of  $C^d$ -families  $(f_a)_a$  of  $C^r$ -diffeomorphisms of  $M$ , so that for a generic  $(f_a)_a \in \mathcal{U}$ , for every parameter  $a \in [-1, 1]^k$ , the map  $f_a$  has infinitely many sinks.*

The same statement is also possible for surface local diffeomorphisms : the dynamics is locally invertible but not globally.

The *main result* of this paper is devoted to get the case  $r \geq d \geq 1$ ,  $r \leq \infty$  and  $d < \infty$ . Hence the cases  $d = r$  or  $r = 1$  are new. It will be stated in section 2.3.

It will be proved thanks to a variation of the previous proof: we will put a source in the covered domain of the parablender. This enables a revision of the previous proof which is shorter, and carries the case  $d = r \geq 1$ .

Also the statement of the main result lies on general hypothesis on an open set of families in order to be useful for a work in progress with S. Crovisier and E. Pujals, showing the Kolmogorov-Arnold  $C^r$ -typicality of dynamics with infinitely many sinks.

Such results are very disturbing since the general trend was to use the bifurcation theory to show the finiteness of attractors. Here the bifurcation theory enables to stop the bifurcation and shows the non-typicality of the finiteness of attractors.

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<sup>3</sup>Similar conjectures had been formulated by Tedeschini-Lalli & Yorke, Palis & Takens, and Palis him-self, some of them for low dimensional dynamical systems.

## 1.2 Emergence

One of my personal motivations is the following problem:

**Problem 9.** *Show the existence of an open set of deterministic dynamical systems which typically cannot be described by means of statistics.*

This problem goes in opposition to the aforementioned optimistic movement, as well as the massive (and naive) use of statistic in many branches of science (economy, ecology, physics ...).

The aim is not to prove that statistics never apply (they do for many systems!), but that they do not apply for many typical systems, even among the finite dimensional, deterministic differentiable dynamical systems. We shall formalize this problem. For this end, we are going to define the Emergence of dynamical systems. This concept evaluates the complexity to approximate a system by statistics.

In statistic it is standard to use the Wasserstein distance  $W_1$  on the space of probability measures  $\mathbb{P}(M)$  of a compact manifold  $M$ :

$$W_1(\nu, \mu) = \sup_{\phi \in Lip^1(M, [-1, 1])} \int_M \phi(x) d(\mu - \nu)(x), \quad \forall \nu, \mu \in \mathbb{P}(M)$$

where  $Lip^1(M, [-1, 1])$  is the space of 1-Lipschitz functions with values in  $[-1, 1]$ .

Given a differentiable map  $f$  of  $M$ ,  $x \in M$  and  $n \geq 0$ , we denote by  $\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)}$  the probability measure which associates to an observable  $\phi \in C^0(M, \mathbb{R})$  the mean  $\frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k(x))$ .

**Proposition 10.** *Given a probability measure  $\mu$ , the following functions are continuous:*

$$x \in M \mapsto d_{W^1}\left(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)}, \mu\right) \in \mathbb{R}, \quad \forall n$$

*Proof.* We notice that it suffices to show that for every  $\delta > 0$ , there exists  $\eta > 0$  such that if  $x$  and  $x'$  are  $\eta$  distant, then

$$d_{W^1}\left(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x')}, \mu\right) \geq d_{W^1}\left(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)}, \mu\right) - \delta.$$

We recall that  $Lip^1(M, [-1, 1])$  endowed with  $C^0$ -uniform norm is compact, by Arzelà-Ascoli Theorem. Hence, there exists  $\phi \in L^1(M, [-1, 1])$  such that:

$$d_{W^1}\left(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)}, \mu\right) = \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k(x)) - \int_M \phi d\mu.$$

As  $\phi$  and  $(f^k)_{k \leq n}$  are Lipschitz, there exists  $\eta > 0$  so that for  $x'$   $\eta$ -close to  $x$ , it holds:

$$\frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k(x')) \geq \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k(x)) - \delta \Rightarrow d_{W^1}\left(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x')}, \mu\right) \geq d_{W^1}\left(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)}, \mu\right) - \delta.$$

□

We recall that the space of probabilities over a compact manifold and endowed with the metric  $d_{W^1}$  is relatively compact.

Hence, given a differentiable map  $f$  of  $M$ , we can define the *Emergence*  $\mathcal{E}(f, \epsilon)$  of  $f$  at scale  $\epsilon > 0$  as the minimum numbers  $N$  of probability measures  $\{\mu_i\}_{1 \leq i \leq N}$  so that

$$\limsup_{n \rightarrow \infty} \int_{x \in M} \min_{1 \leq i \leq N} d_{W^1}\left(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)}, \mu_i\right) d\text{Leb} \leq \epsilon.$$

**Definition 11** (Emergence). *The Emergence is **F** if  $\mathcal{E}(f, \epsilon) = O(1)$  when  $\epsilon \rightarrow 0$ .*

*The Emergence is at most **P** if there exists  $k > 1$  so that  $\mathcal{E}(f, \epsilon) = O(\epsilon^{-k})$ .*

*The Emergence is **Sup-P** if  $\limsup \frac{\log \mathcal{E}(f, \epsilon)}{-\log \epsilon} = +\infty$ .*

We notice that the Emergence is a lower bound on the complexity (in space<sup>4</sup> and in time) to approximate numerically a dynamical system by statistics with precision  $\epsilon$ . Following, the celebrated Cobham's thesis, an algorithm in Sup-P is – in practical – not feasible [Cob65].

Note that the Emergence is invariant by differentiable conjugacy. Also the Emergence of a product of two systems is the product of their Emergences.

**Examples with F-Emergence** If a dynamical system  $f$  admits finitely many ergodic attractor  $(\Lambda_i, \mu_i)_{1 \leq i \leq N}$  whose basins  $(B_i)_i$  cover Lebesgue almost all the manifold, then the Emergence is bounded by  $N$  (and so it is of type **F**)

*Proof.* By the dominated function theorem, it suffices to show that for every  $i \leq N$  and every  $x \in B_i$ ,  $d_{W^1}(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)}, \mu_i) \rightarrow 0$ . By compactity of  $Lip^1(M, [-1, 1])$ , for every  $n$ , there exists  $\phi_n \in Lip^1(M, [-1, 1])$  so that:

$$\Delta_n := d_{W^1}(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)}, \mu_i) = \int_M \frac{1}{n} \sum_{k=0}^{n-1} \phi_n(f^k(x)) d\text{Leb} - \int_M \phi_n d\mu_i.$$

Let  $\phi \in Lip^1(M, [-1, 1])$  be a cluster value of  $(\phi_n)_n$  and let  $(n_j)_{j \geq 0}$  be an increasing sequence so that  $\phi_{n_j} \rightarrow \phi$ . Then

$$\Delta_{n_j} \leq 2 \int_M \|\phi_{n_j} - \phi\|_{C^0} d\mu_i + \int_M \frac{1}{n_j} \sum_{k=0}^{n_j-1} \phi(f^k(x)) d\text{Leb} - \int_M \phi d\mu_i \rightarrow 0.$$

Thus every cluster value of  $(\Delta_n)_n$  is zero, and so this sequence converges to zero.  $\square$

*Remark 12.* We recall that a diffeomorphism satisfying Axiom A, an irrational rotation or a Hénon map for Benedicks-Carleson parameters have finitely many ergodic attractors whose basin cover Lebesgue almost all the phase space  $M$ . Hence their Emergences are finite.

**Example with P-Emergence.** Let  $f$  be the identity. Observe that  $\mathcal{E}(f, \epsilon) = O(\epsilon^{-n})$  with  $n$  the dimension of  $M$ . Hence its Emergence is polynomial. Also the Emergence of an irrational rotation on a cylinder, which is the product of systems with Emergences 1 and  $O(\epsilon^{-1})$ , is  $O(\epsilon^{-1})$ .

It seems also possible to prove that the Emergence of the so-called Bowen eyes dynamics is  $O(\epsilon^{-1})$ .

Hence it seems that all the well understood dynamical systems have an Emergence at most **P**. However, the main conjecture of this work states that those of Sup-P Emergence should not be neglected:

**Conjecture A.** *There exists an open set  $U \subset \text{Diff}(M)$  so that a typical  $f \in U$  has Emergence Sup-P.*

Let us explain why a proof of this conjecture would solve Problem 9 from the computational view point. Given a typical  $f \in U$ , to describe by means of statistics with precision  $\epsilon$ , all of its orbit, but a proportion Lebesgue measure  $1 - \epsilon$ , we would need at least a super-polynomial number of invariant probabilities w.r.t.  $\frac{1}{\epsilon}$ . To find them by means of statistics, we need at

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<sup>4</sup>The number of data to store.

least one data for each of them, and so to do a super polynomial of number of operations. By Cobham's thesis this is not feasible by a computer.

Also we notice that when the Emergence is Sup-P, the Hausdorff dimension of the set of probabilities which would model our system is infinite.

Hence to find these invariant probabilities, we would not be able to use the (finite dimensional) parametric statistics, but only the non-parametric ones, whose computational cost is higher (and much more than 1 as in the above lower bound).

Furthermore let us notice that even if we quotient the phase space by a symmetry group of finite dimension (as for the case of a rotation on the disk or the identity on a manifold), the Emergence of the system will remain Sup-P.

Note that it is not even easy to find a locally typical non-conservative system with infinite Emergence (that is not in  $F$ ), on the other hand KAM theory provides examples (of at least  $P$ -Emergence) in the conservative setting.

**Candidates for Sup-P-Emergence.** It is perhaps possible to construct a unimodal map with Sup-P Emergence from [HK90], or a locally  $C^r$ -dense set of surface diffeomorphisms with Sup-P Emergence from [KS15]. It would be very challenging to derivate from these systems one which is moreover locally typical.

Dynamics with infinitely many sinks do not have finite Emergence. It is perhaps possible to make a variation of Newhouse's construction to produce a generic dynamics with Sup-P Emergence. That is why main Theorem A enters in this program.

Let me mention also the concept of universal dynamics of Bonatti-Diaz [BD02] and Turaev [Tur15] which might produce locally Baire generic sets of diffeomorphism with high Emergence.

It would be interesting to study Conjecture A w.r.t. different notions of typicality [HK10] and smoothness. Also it might be interesting to investigate the concept of Emergence for other metrics than  $W_1$  on the space of invariant probability measures.

Also it would be interesting to provide numerical evidences for such a program (from big data?). The following problem remains open.

**Problem 13.** *Show numerical simulations depicting a (typical) dynamical systems which displays infinitely many sinks.*

Let us point out that by definition, a Sup-P Emergent dynamical system is very complex to describe, and so the non-existence of such pictures is consistent with their conjectured local typicality.

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*With all my thought to my master Jean-Christophe Yoccoz.*

## 2 Statement of the main Theorem

### 2.1 Topological spaces of families of maps

**Space of families** For  $d \geq r \geq 0$  and  $k \geq 0$ , there are at least two ways to define a space of  $C^d$ -families parametrized by  $\mathbb{I}^k := [-1, 1]^k$  of  $C^r$ -maps from a manifold  $M$  into a manifold  $N$ .

The first is attributed to Arnold by Y. Iliachenko (in the case  $d = r$ ) [IL99]. It is the space

$$C_A^{d,r}(\mathbb{I}^k, M, N) := \{(f_a)_a : \partial_a^i \partial_z^j f_a(z) \text{ exists continuously } \forall i \leq d, i+j \leq r \text{ and } (a, z) \in \mathbb{I}^k \times M\}$$

It has the advantage to be invariant by composition: for every  $(f_a)_a, (g_a)_a \in C_A^{d,r}(\mathbb{I}^k, M, M)$ , the composed family  $(f_a \circ g_a)_a$  is in  $C_A^{d,r}(\mathbb{I}^k, M, M)$ .

Another way was presented in [PS96] to state Conjecture 7. It is the space:

$$C_{PS}^{d,r}(\mathbb{I}^k, M, N) := \{(f_a)_a : \partial_a^i \partial_z^j f_a(z) \text{ exists continuously } \forall j \leq r, i \leq d \text{ and } (a, z) \in \mathbb{I}^k \times M\}$$

It has the inconvenient *to not be* invariant by composition when  $d > 0$  and  $r < \infty$ . But it has the advantage to have a geometric meaning. A family  $(f_a)_a \in C_{PS}^{d,r}(\mathbb{I}^k, M, M)$  is actually a  $C^d$ -map from  $\mathbb{I}^k$  into the Fréchet manifold  $C^r(M, N)$ .

We remark:

$$C_A^{d,d+r}(\mathbb{I}^k, M, N) \subset C_{PS}^{d,r}(\mathbb{I}^k, M, N) \subset C_A^{d,r}(\mathbb{I}^k, M, N)$$

Hence in the important case  $r = \infty$  (or  $d = 0$ ) the spaces  $C_{PS}^{d,r}(\mathbb{I}^k, M, N)$  and  $C_A^{d,r}(\mathbb{I}^k, M, N)$  are equal, and so they are denoted by  $C^{d,\infty}(\mathbb{I}^k, M, N)$  (resp.  $C^{0,r}(\mathbb{I}^k, M, N)$ ).

**Topologies on families** Any Riemannian metrics on  $M$  and  $N$ , together with the Euclidean norm on  $\mathbb{R}^k$  define a Riemannian metric on  $N, TM^* \otimes TN, \dots, (\mathbb{R}^{*k})^{\otimes i} \otimes (TM^*)^{\otimes j} \otimes TN$ . The topology of  $C_A^{d,r}(\mathbb{I}^k, M, N)$  is defined thanks to the following base of neighborhoods:

$$V(f, K, \epsilon, r') = \{f' \in C_A^{d,r}(\mathbb{I}^k, M, N) : d(\partial_a^i \partial_z^j f_a(z), \partial_a^i \partial_z^j f'_a(z)) < \epsilon, \forall (a, z) \in K, i+j \leq r', i \leq d\}$$

among any finite  $r' \leq r, \epsilon > 0$  and any compact subset  $K$  of  $\mathbb{I}^k \times M$ . The topology on  $C_{PS}^{d,r}(\mathbb{I}^k, M, N)$  is defined similarly ( $i+j \leq r'$  is replaced by  $j \leq r'$ ). Both topologies coincide for  $r = \infty$  and  $d = 0$ .

We remark that for  $d = r$  the space  $C_A^{d,\infty}(\mathbb{I}^k, M, N)$  is canonically homeomorphic to the space  $C^d(\mathbb{I}^k \times M, N)$  endowed with the  $C^d$ -compact-open topology. Also for  $d = r = \infty$  the space  $C^{\infty,\infty}(\mathbb{I}^k, M, N)$  is canonically homeomorphic to the space  $C^\infty(\mathbb{I}^k \times M, N)$  endowed with the compact-open, weak Whitney topology. A family in  $C^{\infty,\infty}(\mathbb{I}^k, M, N)$  is called *smooth*.

## 2.2 Hyperbolic sets involved

Most of the proofs involve surface local diffeomorphisms. We recall that a map  $f \in C^r(M, M)$  is a local diffeomorphism if  $r \geq 1$  and there is an open covering  $(U_i)_i$  of  $M$  so that  $f|_{U_i}$  is a diffeomorphism onto its image for every  $i$ .

Let us recall some elements of the hyperbolic theory for local diffeomorphisms.

An invariant compact set  $K$  for  $f$  is *hyperbolic* if there is a vector bundle  $E^s \subset TM|_K$  which is invariant by  $Df|_K$ , contracted by  $Df$  and so that the quotient  $TM|_K/E^s$  is expanded by the action induced by  $Df$ .

Then for every  $z \in K$ , the following set, called *stable manifold* of  $z$ , is a  $\dim E^s$ -manifold, injectively  $C^r$ -immersed into  $M$ :

$$W^s(z; f) := \{z' \in M : \lim_{n \rightarrow +\infty} d(f^n(z), f^n(z')) = 0\}$$

The notion of unstable manifold needs to consider the *space of preorbits*  $\overleftarrow{K} := \{(z_i)_{i \leq 0} \in K^{\mathbb{Z}^-} : z_{i+1} = f(z_i) \forall i < 0\}$  of  $K$ . Given a preorbit  $\underline{z} = (z_i)_{i \leq -1} \in \overleftarrow{K}$ , we can define the *unstable*

manifold of  $\underline{z}$ , which is a codim  $E^s$ -manifold  $C^r$ -immersed into  $M$ :

$$W^u(\underline{z}; f) := \{z' \in M : \lim_{n \rightarrow +\infty} d(f^n(z), f^n(z')) = 0\}$$

In general this manifold is *not* immersed *injectively*.

When  $z \in K$  is periodic, the unstable manifold  $W^u(z; f)$  denotes the one associated to the unique preorbit of  $z$  which is periodic.

A local stable manifold  $W_{loc}^s(z; f)$  of  $z$  is an embedded, connected submanifold equal to a neighborhood of  $z$  in  $W^s(z; f)$ . The local unstable manifold are defined similarly. We can chose them so that they depend continuously on  $z$  and  $\underline{z}$  respectively.

We endow  $\overleftarrow{K}$  with the topology induced by the product topology of  $K^{\mathbb{Z}}$ . Hence  $\overleftarrow{K}$  is compact. Note that when  $f|_K$  is bijective,  $\overleftarrow{K}$  is homeomorphic to  $K$ .

**Blender** A hyperbolic set  $K$  of a surface local diffeomorphisms  $f$  is *blender* if  $\dim E^u \neq 2$  and a continuous union of local unstable manifolds  $\cup_{z \in K} W_{loc}^u(z; f)$  contains robustly a non empty open set  $O$  of  $M$ :

$$\cup_{z \in K} W_{loc}^u(z; f') \supset O \quad , \quad \forall f' \text{ } C^1\text{-close to } f \text{ .}$$

The set  $O$  is called a *covered domain* of the blender  $K$ .

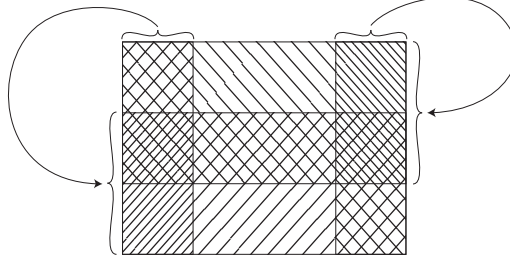


Figure 1: A blender of a surface.

Blender were discovered in [BD96], and then used in [BD99, DNP06] to produce a locally generic set of diffeomorphisms displaying infinitely many sinks. In [Ber16b], the notion of blender has been adapted to local surface diffeomorphisms to produce a locally generic set of surface local diffeomorphism displaying infinitely many sinks following a similar argument to [DNP06].

**Area contracting saddle point** A surface local-diffeomorphism has an area contracting fixed point  $P$  if the product of the stable and the unstable eigenvalues of  $P$  has a modulus less than 1.

**Projectively hyperbolic source** A fixed point  $S$  of a surface diffeomorphism  $f$  is a *projectively hyperbolic source* if  $D_S f$  has two eigenvalues  $\sigma_{uu}, \sigma_u$  with different moduli  $1 < |\sigma_u| < |\sigma_{uu}|$ . The eigenspace associated to  $\sigma_u$  is called the *weak unstable direction*, whereas the eigenspace  $E^{uu}(S)$  associated to  $\sigma_{uu}$  is called the *strong unstable direction*.

A *basin* of the source  $S$  is an open neighborhood  $B$  of  $S$  on which an inverse branch  $g$  of  $f$  is well defined and whose points  $z \in B$  satisfy  $g^n(z) \rightarrow S$ . Then  $E^{uu}(S)$  extends continuously



to a line field on  $B$ , denoted also by  $E^{uu}(S)$  and so that for every  $z \in N$ ,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log(\|D_z g^n|E^{uu}(S)\| \cdot |\sigma^{uu}|^n) \rightarrow 0.$$

The line field is uniquely defined once  $g$  is fixed, and there is a unique inverse branch of  $g: B \rightarrow B$  which fixes  $S$ . Hence  $\mathcal{F}^{uu}(S)$  and  $E^{uu}(S)$  are uniquely defined once  $S$  and  $B$  are fixed.

If  $S$  is a source of period  $p$ , then the above definitions and notations are canonically generalized by considering  $f^p$  instead of  $f$ .

Moreover it is well known that the line field  $E^{uu}$  is the tangent space of a unique  $C^0$ -foliation  $\mathcal{F}^{uu}$  on  $N$ , whose leaves are as regular as the dynamics [Yoc95].

A  $C^1$ -embedded curve  $\Gamma$  in  $B$  has a *robust tangency* with  $\mathcal{F}^{uu}$  if any  $C^1$ -perturbation of  $\Gamma$  has a tangency with one leaf of  $\mathcal{F}^{uu}$ .

**Hyperbolic sets for families of dynamics** Let us fix  $k \geq 0$ ,  $1 \leq d \leq r \leq \infty$ ,  $X \in \{A, PS\}$  and a family of local diffeomorphisms  $(f_a)_a \in C_X^{d,r}(\mathbb{I}^k, M, M)$ , with  $\mathbb{I} = [-1, 1]$ .

It is well known that if  $f_0$  has a hyperbolic fixed point  $P_0$ , then it persists for every  $a$  small as a hyperbolic fixed point  $P_a$ , and the map  $a \mapsto P_a$  is of class  $C^d$ .

More generally, if  $K$  is a hyperbolic set for  $f_0$ , it persists for every  $a$  small, but if the map  $f_0|_K$  is not bijective, we need to consider the space of preorbits  $\overleftarrow{K} = \{(z_i)_{i \leq 0} : f_0(z_i) = z_{i+1} \forall i < 0\}$  of  $K$ . Let  $\overleftarrow{f}_0$  be the shift map on  $\overleftarrow{K}$ .

**Theorem 14** (Prop 1.6 [Ber16b, Ber16a]). *For every  $a$  in a neighborhood  $V$  of 0, there exists a map  $h_a \in C^0(\overleftarrow{K}; M)$  so that:*

- $h_0$  is the zero-coordinate projection  $(z_i)_i \mapsto z_0$ .
- $f_a \circ h_a = h_a \circ \overleftarrow{f}_0$  for every  $a \in V$ .
- For every  $\underline{z} \in \overleftarrow{K}$ , the map  $a \in V \mapsto h_a(\underline{z})$  is of class  $C^d$ .

The point  $h_a(\underline{z})$  is called the *hyperbolic continuation* of  $\underline{z}$  for  $f_a$ . We denote  $\underline{z}_a \in M$  the zero-coordinate of  $h_a(\underline{z})$ . The family of sets  $(K_a)_a$ , with  $K_a := \{\underline{z}_a : \underline{z} \in \overleftarrow{K}\}$ , is called the hyperbolic continuation of  $K$ .

The local stable and unstable manifolds  $W_{loc}^s(z; f_a)$  and  $W_{loc}^u(\underline{z}; f_a)$  are canonically chosen so that they depend continuously on  $a$ ,  $z$  and  $\underline{z}$ . They are called the *hyperbolic continuations* of  $W_{loc}^s(z; f)$  and  $W_{loc}^u(\underline{z}; f)$  for  $f_a$ . Let us recall:

**Proposition 15** (Prop 1.6 [Ber16b, Ber16a]). *For every  $z \in K$ , the family  $(W_{loc}^s(z; f_a))_{a \in V}$  is of class  $C_A^{d,r}$ . For every  $\underline{z} \in \overleftarrow{K}$ , the family  $(W_{loc}^u(\underline{z}; f_a))_{a \in V}$  is of class  $C_A^{d,r}$ . Both vary continuously with  $z \in K$  and  $\underline{z} \in \overleftarrow{K}$ .*

The bifurcation theory is the branch of dynamical systems which studies the hyperbolic continuation of hyperbolic sets and their local stable and unstable manifolds, to find dynamical properties.

Hence it is natural to study the action of  $(f_a)_a$  on  $C^d$ -jets. Given a  $C^d$ -family of points  $(z_a)_{a \in \mathbb{R}^k}$ , its  $C^d$ -jet at  $a_0 \in \mathbb{R}^k$  is  $J_{a_0}^d(z_a)_a = \sum_{j=0}^d \frac{\partial_a^j z_a}{j!} a^{\otimes j}$ . Let  $J_{a_0}^d(\mathbb{R}^k, M)$  be the space of  $C^d$ -jets of  $C^d$ -families of points in  $M$ .

We notice that any  $C_A^{d,r}$ -family  $(f_a)_a$  of  $C^r$ -maps  $f_a$  of  $M$  acts canonically on  $J_{a_0}^d M$  as the map:

$$J_{a_0}^d(f_a)_a : J_{a_0}^d(z_a)_a \in J_{a_0}^d(\mathbb{R}^k, M) \mapsto J_{a_0}^d(f_a(z_a))_a \in J_{a_0}^d(\mathbb{R}^k, M)$$

*Remark 16.* Suppose that  $M$  is a surface. If  $f_{a_0}$  has a hyperbolic fixed point  $P$  with stable and unstable eigenvalues  $\lambda_s, \lambda_u$  then  $J_{a_0}^d(P)_a$  is the unique hyperbolic fixed point of  $J_{a_0}^d(f_a)_a$ . Moreover the stable and unstable directions of  $D_{J_{a_0}^d(P)_a} J_{a_0}^d(f_a)_a$  have the same dimension. The restriction of  $D_{J_{a_0}^d(P)_a} J_{a_0}^d(f_a)_a$  to each of these spaces is the composition of  $\lambda_s id$  (resp.  $\lambda_u id$ ) with a nilpotent map. We observe that  $W^s(J_{a_0}^d(P)_a)$  consists of  $C^d$ -jets of families  $(Q_a)_a$  so that  $Q_a$  is in  $W^s(P_a; f_a)$  for every  $a$ .

More generally, given a hyperbolic set  $K$  for  $f_{a_0}$ , the set  $J_{a_0}^d(K)_a := \{J_{a_0}^d(h_a(\underline{z}))_a : \underline{z} \in \widehat{K}\}$  is a hyperbolic compact set for  $(J_{a_0}^d(f_a))_a$ .

The first example of parablender was given in [Ber16b]; in [BCP16] a new example of parablender was given and the concept reaches the following definition:

**Definition 17** ( $C^d$ -Parablender). *A family  $(K_a)_a$  of blenders for  $(f_a)_a$  is a  $C^d$ -parablender at  $a = a_0$  if the following condition is satisfied. There exists a non-empty open set  $O$  of  $C^d(\mathbb{R}^k, M)$  so that for every  $(f'_a)_a$   $C_A^{d,d}$ -close to  $(f_a)_a$ , for every  $\gamma \in O$ , there exist  $\underline{z} \in \widehat{K}$  and a  $C^d$ -curve of points  $(Q_a)_a$  in a continuous family of local unstable manifolds  $(W_{loc}^u(\underline{z}; f'_a))_a$  satisfying:*

$$d(\gamma(a), Q_a) = o(\|a - a_0\|^d).$$

*The open set  $O$  is called a covered domain for the  $C^d$ -parablender  $(K_a)_a$ .*

*Remark 18.* We notice that if  $J^d(K_a)_a$  is a blender for  $J_{a_0}^d(f_a)_a$  then  $(K_a)_a$  is a  $C^d$ -parablender at  $a_0$  for  $(f_a)_a$ . We do not know if it is a necessary condition.

*Example 19* ( $C^d$ -Parablender). We propose here a small variation of Example 2.2 [Ber16b]. Let  $\Delta := \{-1, 1\}^E$  with  $E := \{i = (i_1, \dots, i_k) \in \{0, \dots, d\}^k : i_1 + \dots + i_k \leq d\}$ .

Consider *Card*  $\Delta$  disjoint segments  $D := \sqcup_{a \in \Delta} I_\delta$  of  $(1, 1) \setminus \{0\}$ . Let  $Q : \sqcup_{\delta \in \Delta} I_\delta \rightarrow [1, 1]$  be a locally affine, orientation preserving map which sends each  $I_\delta$  onto  $[-1, 1]$ . Let  $(\mathring{f}_a)_a$  be the  $k$ -parameters family defined by:

$$\mathring{f}_a(x, y) : (x, y) \in D \times [-3, 3] \longmapsto (Q(x), \frac{2}{3}y + \sum_{i \in E} \delta(i) \cdot a_1^{i_1} \cdots a_k^{i_k}) \quad \text{if } x \in I_\delta.$$

We notice that the maximal invariant set of  $\mathring{f}_0$  is a blender  $K$ .

Let us define the following subset of  $J_0^d \mathbb{R}^2$ :

$$\hat{O} := \left\{ \sum_{i \in E} (x_i, y_i) \cdot a_1^{i_1} \cdots a_k^{i_k} : |x_i| < 1, |y_i| < 2 \right\}$$

$$\hat{O}_\delta := \left\{ \sum_{i \in E} (x_i, y_i) \cdot a_1^{i_1} \cdots a_k^{i_k} : |x_i| < 1, 0 \leq \delta(i) \cdot y_i < 2 \right\}$$

We observe that  $\hat{O} = \cup_{\delta \in \Delta} \hat{O}_\delta$ . Also for every  $\delta \in \Delta$ ,  $J_0^d(\mathring{f}_a)_a$  maps  $\hat{O}_\delta$  into the compact set:

$$\hat{O}' := \left\{ \sum_{i \in E} (x_i, y_i) \cdot a_1^{i_1} \cdots a_k^{i_k} : |x_i| \leq 1/2, |y_i| \leq 1 \right\} \Subset \mathcal{O}.$$

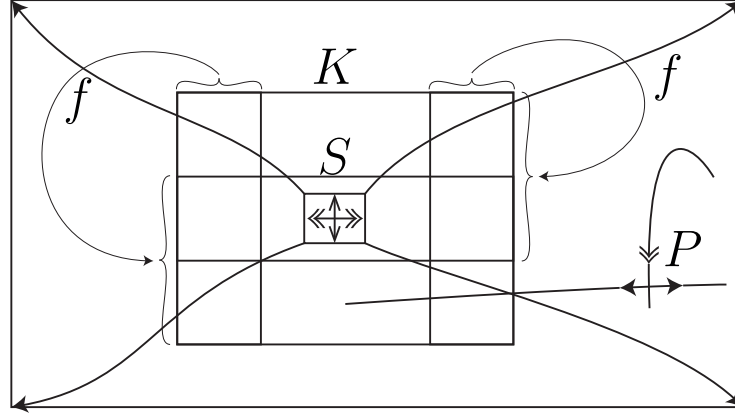
By proceeding like in Theorem B [BCP16], it comes that the hyperbolic continuity  $(\mathring{K}_a)_a$  of  $K$  is a  $C^d$ -parablender at  $a_0 = 0$  with  $\hat{O}$  in its covered domain. Hence there exists  $\alpha > 0$  small such that it is a  $C^d$ -parablender at every  $\|a_0\| \leq \sqrt[k]{2}\alpha$ .

Consequently the family map  $(f_a)_a := (\mathring{f}_{\alpha \cdot a})_a$  has the  $C^d$ -parablender  $(K_a)_a = (\mathring{K}_{\alpha \cdot a})_a$  at every  $a_0 \in \mathbb{I}^k$ , with covered domain containing every constant family of points in  $[-1/2, 1/2] \times [-1, 1]$ .

## 2.3 Statement of the main Theorem

Let  $\mathcal{U}$  be the open set of  $C^1$ -local surface diffeomorphisms of a surface  $M$  which have a blender  $K$ , an area contracting saddle fixed point  $P$  and a projectively hyperbolic source  $S$  with a strong unstable foliation  $(\mathcal{F}^{uu}, B)$  so that :

- ( $H_0$ )  $S$  is in a domain robustly covered by a continuous family  $(W_{loc}^u(\underline{z}; f_a))_{\underline{z} \in \tilde{K}}$  of local unstable manifolds of the blender  $K$ , which are not tangent to the weak unstable direction of  $S$ .
- ( $H_1$ )  $K$  is included in  $B$  and the stable direction of  $K$  is not tangent to  $\mathcal{F}^{uu}$ .
- ( $H_2$ ) A segment of  $W^s(P; f)$  has a robust tangency with  $\mathcal{F}^{uu}$  and  $W^u(P; f)$  has a transverse intersection with  $W^s(K; f)$ .



It will be clear after the reading of the sketch of proof of the main theorem that a  $C^r$ -generic diffeomorphism in  $\mathcal{U}$  displays infinitely many sinks for every  $\infty \geq r \geq 1$ . See also [New80, Asa08] for a parameter free argument.

Actually if there is a  $C^{d,d}$ -families  $(f_a)_a$  of maps  $f_a$  in  $\mathcal{U}$  so that the hyperbolic continuation  $(K_a)_a$  of  $K$  is a  $C^d$ -parablender which contains the hyperbolic continuation  $(S_a)_a$  of  $S$  in its covered domain at every  $a_0 \in \mathbb{I}^k = [-1, 1]^k$ , then this phenomena holds in a stronger sens.

More precisely, given  $k \geq 0$ , for every  $\infty > d \geq 1$ , let  $\mathcal{U}_A^{d,d}$  be a  $C_A^{d,d}$ -open set of  $k$ -parameters families  $(f_a)_{a \in \mathbb{I}^k}$  of local diffeomorphisms  $f_a \in \mathcal{U}$ , so that for every  $a_0$ , there exists  $\underline{z} \in \tilde{K}$  and a  $C^d$ -curve of points  $(Q_a)_a$  in the continuation  $(W^u(\underline{z}; f_a))_a$  of  $W_{loc}^u(\underline{z}; f)$  satisfying:

- ( $H_3$ )  $d(Q_a, S_a) = o(\|a - a_0\|^d)$ , in particular  $W_{loc}^u(\underline{z}; f_{a_0})$  contains  $S_{a_0}$ ,
- ( $H_4$ )  $W_{loc}^u(\underline{z}; f_{a_0})$  is not tangent to the weak unstable direction of  $S_{a_0}$ .

In particular  $(K_a)_a$  is a  $C^d$ -parablender and the source  $(S_a)_a$  belongs to the covered domain of the  $C^d$ -parablender  $(K_a)_a$  at every  $a_0$ . Note that ( $H_3$ ) & ( $H_4$ ) imply  $H_0$ .

We observe that for every  $X \in \{A, PS\}$ , let  $1 \leq d \leq r \leq \infty$ , the set  $\mathcal{U}_X^{d,r} := \mathcal{U}_A^{d,r} \cap C_X^{d,r}(\mathbb{I}^k, M, M)$  is open for the  $C_X^{d,r}$ -topology. We recall that  $\mathbb{I}^k := [-1, 1]^k$ .

**Theorem A** (Main theorem). *For every  $1 \leq k < \infty$ , any topology  $X \in \{A, PS\}$ ,  $1 \leq d \leq r \leq \infty$  with  $d < \infty$ , there exists a  $C_X^{d,r}$ -Baire generic set  $\mathcal{R}$  in  $\mathcal{U}_X^{d,r}$  so that for every  $(f_a)_a \in \mathcal{R}$  and every  $a \in \mathbb{I}^k$ , the map  $f_a$  displays infinitely many sinks.*

*Example 20.* Here is a variation of §4 of [Ber16b] in which we add a source.

Let  $(f_a)_{a \in \mathbb{I}^k} : D \times [-3, 3] \rightarrow [-1, 1] \times [-4, 4]$  be the map of example 19 exhibiting a  $C^d$ -parablender  $(K_a)_a$  at every  $a \in \mathbb{I}^k$  and the constant family  $(0)_{a \in \mathbb{I}^k}$  in its covered domain. We

recall that  $D$  is made by  $Card \Delta$ -intervals of  $(-1, 1) \setminus \{0\}$ . Let  $I_S, I_P, I_{P'} \subset (-1, 1) \setminus D$  be disjoint segments, so that  $I_S$  is centered at 0. We extend  $Q$  to  $D \sqcup I_S \sqcup I_P \sqcup I_{P'}$  so that  $Q$  remains locally affine and orientation preserving, and sends as well  $I_S, I_P, I_{P'}$  onto  $[1, 1]$ . Let  $x_S \in I_S$  and  $x_P$  be fixed points of  $Q$ . Let  $x_{P'}$  be the preimage by  $Q|_{I_{P'}}$  of  $x_P$ . Let:

$$f_a(x, y) : (x, y) \in (D \sqcup I_S \sqcup I_P \sqcup I_{P'}) \times [-3, 3] \mapsto \begin{cases} f_a(x, y) & \text{if } x \in D \\ (Q(x), \frac{y}{\sqrt{|I_S|}}) & \text{if } x \in I_S, \\ (Q(x), |I_P|^2 y) & \text{if } x \in I_P, \\ (y - (x - x_{P'})^2 + x_P, x_{P'} - x) & \text{if } x \in I_{P'}, \end{cases}$$

With  $\hat{\mathbb{R}}$  the one point compactification of  $\mathbb{R}$ , since  $Q$  is orientation preserving, it is easy to extend  $f$  to a local diffeomorphism of the torus  $\hat{\mathbb{R}}^2$  of degree  $Card \Delta + 3$ .

We notice that  $S = (0, 0)$  is a projectively hyperbolic source  $S$  with vertical weak unstable direction, hence transverse the local unstable manifold of  $(K_a)_a$  (which are of the form  $([-1, 1] \times \{y_a\})_a$ ).

Note also that the hyperbolic continuation of  $S$  is the constant family  $(0)_a$  and so belongs to the covered domain of the  $C^d$ -parablender  $(K_a)_a$  at every  $a_0 \in \mathbb{I}^k$ .

Also  $P = (x_P, 0)$  is an area contracting saddle fixed point, with vertical local stable manifold. The preimage of this local stable manifold in  $I_{P'} \times [-1, 1]$  is the graph of the function  $x \mapsto (x - x_{P'})^2$ . It has a robust tangency with  $\mathcal{F}^{uu}(S)$  whose leaves are all horizontal.

Consequently  $(f_a)_{a \in \mathbb{I}^k}$  belongs to  $\mathcal{U}_A^{d,d} \cap C^{\infty, \infty}(\mathbb{I}^k, M, M)$ .

Hence by Theorem A, for every  $\infty \geq r \geq d$  and  $X \in \{A, PS\}$ , a  $C_X^{d,r}$ -Baire generic perturbation of  $(f_a)_a$  displays infinitely many sinks at every parameter  $a \in \mathbb{I}^k$ .

A corollary of the above example and of the proof of the Main theorem is:

**Corollary B.** *For every compact manifold of dimension  $\geq 3$ , for all  $\infty > r \geq d \geq 1$ ,  $\infty > k \geq 0$ , and  $X \in \{PS, A\}$ , there exists an open set  $\hat{U}$  in  $C_X^{d,r}(\mathbb{I}^k, M, M)$  of families  $(\hat{f}_a)_a$  of diffeomorphisms  $\hat{f}_a \in Diff^r(M)$  and a Baire residual set  $\mathcal{R}$  in  $\hat{U}$  so that for every  $(f_a)_a \in \mathcal{R}$ , for every  $a \in \mathbb{I}^k$ , the map  $f_a$  displays infinitely many sinks.*

The proof will be done in section 8.

### 3 Sketch of proof

Let  $k \geq 1$ ,  $r \geq d \geq 1$  with  $d < \infty$  and  $X \in \{A, PS\}$ . To avoid technical difficulties, we will work only with  $C^\infty$ -families in  $\mathcal{U}_X^{d,r}$ , which are families in  $\mathcal{U}^\infty$ , where:

$$\mathcal{U}^\infty := \mathcal{U}_A^{d,d} \cap C^{\infty, \infty}(\mathbb{I}^k, M, M) = \mathcal{U}_X^{d,r} \cap C^{\infty, \infty}(\mathbb{I}^k, M, M), \quad \mathbb{I}^k = [-1, 1]^k$$

We observe that  $\mathcal{U}^\infty$  is a dense set in  $\mathcal{U}_X^{d,r}$  for the  $C_X^{d,r}$ -topology. It is also dense in the following space:

$$\mathcal{U}^{d, \infty} := \mathcal{U}_A^{d,d} \cap C^{d, \infty}(\mathbb{I}^k, M, M) = \mathcal{U}_X^{d,r} \cap C^{d, \infty}(\mathbb{I}^k, M, M)$$

The following lemma enables us to work only with the spaces  $\mathcal{U}^{d, \infty}$  and  $\mathcal{U}^\infty$ .

**Lemma 21.** *The main Theorem holds true if for every  $M > 0$ , there is a dense set in  $\mathcal{U}^{d, \infty}$  of families  $(f_a)_a \in \mathcal{U}^\infty$ , so that for every  $a \in \mathbb{I}^k$ , the map  $f_a$  has a sink of period at least  $M$ .*

*Proof.* For every  $M \geq 1$ , the set  $\mathcal{V}_{M,A}$  of families  $(f_a)_a \in \mathcal{U}_X^{d,r}$  such that  $f_a$  has a sink of period at least  $M$  for every  $a \in \mathbb{I}^k$  is open and dense in  $\mathcal{U}_X^{d,r}$ . Hence the following set is Baire residual in  $\mathcal{U}_X^{d,r}$ :  $\mathcal{R} := \bigcap_{M \in \mathbb{N}} \mathcal{V}_M$ . We observe that for every  $(f_a)_a \in \mathcal{R}$ , for every  $a \in \mathbb{I}^k$ , the map  $f_a$  has a sink of arbitrarily large period. Hence  $f_a$  has infinitely many sinks.  $\square$

We recall that for any  $(f_a)_a \in \mathcal{U}^\infty$  and any  $a_0 \in \mathbb{I}^k$  the source  $(S_a)_a$  has its  $C^d$ -jets at  $a = a_0$  in the covered domain of the  $C^d$ -parablender  $(K_a)_a$ . This means that there exists  $\overleftarrow{Q} \in \overleftarrow{K}$  and a  $C^\infty$ -curve of points  $(Q'_a)_a$  in the local unstable manifold  $(W_{loc}^u(\overleftarrow{Q}; f_a))_a$  so that  $d(Q'_a, S_a) = o(\|a - a_0\|^d)$ . Hence a  $C^{d,\infty}$ -perturbation of  $(f_a)_a$  put  $S_a$  at  $Q'_a$  without changing the unstable manifold of  $Q_a$  for all  $a$  in a neighborhood of  $a_0$ .

Moreover, by  $(H_2)$ , the unstable manifold of  $P$  has a transverse intersection with a stable manifold of  $K$ . Hence by the following parametrized inclination lemma,  $(W^u(P; f_a))_a$  accumulates on  $(W_{loc}^u(\overleftarrow{Q}; f_a))_a$ .

**Lemma 22** (Parametrized inclination Lemma 1.7 [Ber16b]). *Let  $(f_a)_a$  be a smooth family of local diffeomorphisms leaving invariant a hyperbolic compact set  $(K_a)_a$  with unstable direction of dimension  $d_u$ . Let  $(C_a)_a$  be a smooth family of submanifolds of dimension  $d_u$  and intersecting transversally a local stable manifold of  $K$ .*

*Then, for any local unstable manifold  $(W_{loc}^u(\overleftarrow{Q}; f_a))_a$  with  $\overleftarrow{Q} \in \overleftarrow{K}$ , for every  $n$ , there exists  $C'_a \subset C_a$  so that the family  $(f_a^n(C'_a))_a$  is  $C^\infty$ -close to  $(W_{loc}^u(\overleftarrow{Q}; f_a))_a$  when  $n$  is large.*

By  $(H_2)$ , this implies that for every  $a_0 \in \mathbb{I}^k$ , there exists a dense set in  $\mathcal{U}^{d,\infty}$  of smooth families  $(f_a)_a \in \mathcal{U}^\infty$  so that for all  $a$  in a neighborhood of  $a_0$ , a segment  $\Gamma_a^u$  of  $W^u(P; f_a)$  contains of  $S_a$ .

Moreover, by  $(H_4)$ , we can assume that the weak unstable direction of  $S_a$  is not tangent to  $\Gamma_a^u$  for every  $a$  close to  $a_0$ .

By  $(H_2)$  we can perturb  $(f_a)_a$  so that a segment of  $W^s(P; f_{a_0})$  has a quadratic tangency with a leaf of  $\mathcal{F}^{uu}(S_{a_0})$ . We recall that two curves of a surface have a *quadratic tangency* if they are tangent, and their curvatures are different at a tangency point.

This is will be useful to have  $(f_a)_a$  with a persistent homoclinic tangency.

**Definition 23** (Persistent homoclinic tangency). *Let  $(f_a)_{a \in \mathbb{I}^k}$  be a smooth family of surface local-diffeomorphisms and  $(P_a)_{a \in \mathbb{I}^k}$  a saddle periodic point.*

*The saddle point  $P_a$  has a homoclinic tangency if  $W^u(P_a; f_a)$  is tangent to  $W^s(P_a; f_a)$  at one point  $H_a$ .*

*The homoclinic tangency is persistent for  $a$  in an open subset  $V \subset \mathbb{I}^k$ , if there exist a smooth family  $(\Gamma_a^u)_{a \in V}$  of embedded segments in  $(W^u(P; f_a))_a$  and a smooth family of points  $(H_a)_{a \in V} \in (\Gamma_a^u)_{a \in V}$  so that  $W^s(P; f_a)$  is tangent to  $\Gamma_a^u$  at  $H_a$  for every  $a \in V$ .*

Consequently, the following proposition implies that for every  $a_0 \in \mathbb{I}^k$ , there exists a dense set in  $\mathcal{U}^{d,\infty}$  of families  $(f_a)_a \in \mathcal{U}^\infty$  so that  $P$  has a persistent homoclinic tangency for a neighborhood of  $a_0$ .

**Proposition 24.** *Let  $V \subset \mathbb{I}^k$  be a compact subset. Let  $(f_a)_{a \in \mathbb{I}^k}$  be a  $C^\infty$ -family of diffeomorphisms, which has a projectively hyperbolic source  $(S_a)_a$ . Let  $(C_a)_a$  be a smooth family of embedded curves  $C_a$ , so that for every  $a \in V$ ,  $C_a$  has a quadratic tangency with  $\mathcal{F}^{uu}(S_a)$  at a point  $c_a$  depending continuously on  $a \in V$ . Let  $(W_a)_a$  be a smooth family of embedded curves so that for every  $a \in V$ ,  $W_a$  contains  $S_a$  in its interior and  $T_{S_a}W_a$  is not equal to the weak unstable direction of  $S_a$ .*

Then there exists a smooth perturbation of  $(W'_a)_a$  of  $(W_a)_a$  and  $n \geq 0$  so that  $f_a^n(W_a)$  has a quadratic tangency with  $C_a$  which persists for  $a \in V$ .

This proposition will be proved in section 4.

Furthermore, the following proposition implies that for every  $M \geq 0$ , for every  $a_0 \in \mathbb{I}^k$ , there exists a dense set in  $\mathcal{U}^{d,\infty}$  of smooth families  $(f_a)_a \in \mathcal{U}^\infty$  so that  $f_a$  has a sinks of period at least  $M$  for every  $a$  in  $V$ .

**Proposition 25.** *Let  $V \subset \mathbb{I}^k$  be a compact subset. Let  $(f_a)_{a \in \mathbb{I}^k}$  be a  $C^\infty$ -family of local diffeomorphisms. We suppose that for every  $a \in V$ , the map  $f_a$  has an area contracting saddle point  $P_a$  which displays a persistent homoclinic tangency at  $H_a$  for  $a \in V$ . Then for every  $M \geq 1$  and  $\eta > 0$ , there exists a smooth perturbation  $(f'_a)_a$  such that for every  $a \in V$  :*

- for every  $z \notin B(H_a, \eta)$ , it holds  $f_a(z) = f'_a(z)$ ,
- the map  $f_a$  has a sink of period at least  $\geq M$ .

This proposition will be proved in section 5.

*Remark 26.* Now it should be clear for the reader that a Baire generic  $C^r$ -map in  $\mathcal{U}$  has infinitely many sinks.

By Lemma 21, to prove the main theorem, it suffices to construct for every  $M \geq 0$ , a dense set in  $\mathcal{U}^{d,\infty}$  of smooth families which have a sink of period  $\geq M$  for every  $a \in \mathbb{I}^k$ .

However, the above argument could not give more (after developing it carefully) that there exists a smooth families  $(f'_a)_a$  which is  $C^{d,\infty}$ -close to  $(f_a)_a$  and  $\epsilon > 0$  arbitrarily small, so that  $f'_a$  has a sink for every parameter parameter  $a$  a  $\epsilon$ -dense open set.

Nevertheless, to apply Lemma 21, we want such a property for *every* parameter and not an  $\epsilon$ -dense set. To get every parameter, we will replicate the source by the following proposition proved in section 7.

**Proposition 27.** *There is a dense set in  $\mathcal{U}^{d,\infty}$  formed by families  $(f_a)_a \in \mathcal{U}^\infty$  which satisfies the following property:*

*There exists a finite open covering  $(U_i)_i$  of  $\mathbb{I}^k$  so that for every  $i$  there exist a projectively hyperbolic source  $(S_{ia})_{a \in U_i}$  and a continuation of embedded segments  $(\Gamma_{ia}^u)_a$  into  $(W^u(P; f_a))_a$  such that:*

- (i)  $S_{ia}$  is in  $\Gamma_{ia}^u \subset W^u(P_a; f_a)$ , for every  $a \in U_i$ .
- (ii)  $T_{S_{ia}} \Gamma_{ia}^u$  is not the weak unstable direction of  $S_{ia}$ , for every  $a \in U_i$ .
- (iii) There exists a (smooth) curve of points  $(H_{ia})_{a \in U_i}$  in  $(W^s(P_a; f_a))_{a \in U_i}$  at which  $W^s(P_a; f_a)$  has a quadratic tangency with the strong unstable foliation of  $S_{ia}$ , for every  $a \in U_i$ .
- (iv) For every  $a \in U_i \cap U_j$  with  $i \neq j$ , the sets  $(f_a^k(H_{ia}))_{k \geq 0}$  and  $(f_a^k(H_{ja}))_{k \geq 0}$  are disjoint and the orbits  $(f_a^k(S_{ia}))_{k \geq 0}$  and  $(f_a^k(S_{ja}))_{k \geq 0}$  are disjoint.

In section 6, a development of the above sketched argument will prove that this proposition implies that for every  $M \geq 0$ , there exists a dense set in  $\mathcal{U}^{d,\infty}$  of smooth families in  $\mathcal{U}^\infty$  which have a sink of period  $\geq M$  for every  $a \in \mathbb{I}^k$ . Then Lemma 21 implies the main theorem.

## 4 Tangency creation (Proof of Prop. 24)

In this section we consider a  $C^\infty$ -family  $(f_a)_a$  of diffeomorphisms of  $\mathbb{R}^2$  which display a projectively hyperbolic source  $S_a$  for every  $a$  with strong unstable direction  $E_a^{uu}$ .

It is useful to regard the following smooth bundle automorphism over  $f_a$ :

$$Tf_a: \mathbb{R}^2 \times \mathbb{P}\mathbb{R}^1 \rightarrow \mathbb{R}^2 \times \mathbb{P}\mathbb{R}^1$$

$$(z, L) \mapsto (f_a(z), D_z f_a(L))$$

We notice that the strong unstable direction  $E_a^{uu}$  is a hyperbolic point for  $Tf_a$  with unstable direction  $\mathbb{R}^2$  and stable direction  $T_{E_a^{uu}}\mathbb{P}\mathbb{R}^1$ .

The following well known proposition is important:

**Proposition 28.** *The strong unstable foliation  $\mathcal{F}^{uu}(S_a)$  on the neighborhood of  $S_a$  is of class  $C^\infty$  and depends  $C^\infty$  on  $a \in \mathbb{I}^k$ .*

*Proof.* Observe that  $(S_a, E^{uu}(S_a))$  is a hyperbolic point of  $Tf_a$  and that the tangent space of  $\mathcal{F}^{uu}(S_a)$  is its a local unstable manifold. As  $Tf_a$  is of class  $C^\infty$ , its local unstable manifold and so the foliation  $\mathcal{F}^{uu}(S_a)$  are of class  $C^\infty$ . As the local stable manifolds of a hyperbolic point of a smooth family of diffeomorphisms depends smoothly on the parameter by Prop. 15, the foliation  $\mathcal{F}^{uu}(S_a)$  depends smoothly on  $a$ .  $\square$

We observe that Proposition 24 is a consequence of the following:

**Proposition 29.** *Let  $(C_a)_a$  be a  $C^\infty$ -family of embedded curves  $C_a$ , and  $V$  a compact set of  $\mathbb{I}^k$ . Assume that for every  $a \in V$ , the curve  $C_a$  has a quadratic tangency with  $\mathcal{F}^{uu}(S_a)$  at a point  $H_a$  depending continuously on  $a$ .*

*Let  $(W_a)_a$  be a  $C^\infty$ -family of embedded curves  $W_a$  which contains  $S_a$  for every  $a$  and so that  $T_{S_a}W_a$  is not equal to the weak unstable direction of  $S_a$ .*

*Then for every compact set  $V$ , there exists a  $C^\infty$ -perturbation of  $(W'_a)_a$  of  $(W_a)_a$  and  $n \geq 0$  so that  $f_a^n(W_a)$  has a quadratic tangency with  $C_a$  for every  $a \in V$ .*

*Proof.* We continue with the above notion by denoting  $Tf_a: \mathbb{R}^2 \times \mathbb{P}\mathbb{R}^1 \rightarrow \mathbb{R}^2 \times \mathbb{P}\mathbb{R}^1$  the action of  $(f_a)_a$  on the line bundle of  $\mathbb{R}^2$ . Let  $E_a^{uu} \in \{S_a\} \times \mathbb{P}\mathbb{R}^1$  is the strong unstable direction of  $S_a$ .

We observe that the tangent bundle  $TC_a$  of  $C_a$  is an embedded curve  $TC_a$  in  $\mathbb{R}^2 \times \mathbb{P}\mathbb{R}^1$ . As  $C_a$  is tangent at one point to  $\mathcal{F}^{uu}(S_a)$ , the curve  $TC_a$  intersects  $W_{loc}^u(E_a^{uu}; Tf_a)$ . As the tangency is quadratic, this intersection is transverse.

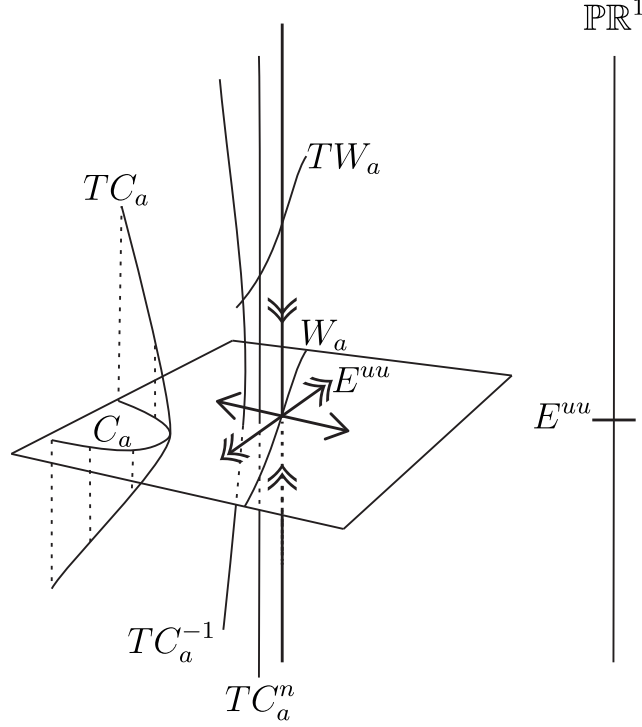
Hence by the inclination lemma, the preimages  $(TC_a^n)_{n \leq -1}$  of  $TC_a$  by  $Tf_a$  accumulate on  $W_{loc}^s(E_a^{uu}; Tf_a)$ , for the  $C^\infty$ -topology. By the parametric inclination lemma 22, the preimages  $((TC_a^n)_a)_{n \leq -1}$  of  $TC_a$  by  $(Tf_a)_a$  accumulate on  $(W_{loc}^s(E_a^{uu}; Tf_a))_a$ , for the  $C^\infty$ -topology.

Remark that  $W^s(E_a^{uu}, Tf_a)$  is  $\{S_a\} \times \mathbb{P}\mathbb{R}^1$  without the weak unstable direction of  $S_a$ .

Note also that the tangent bundle  $TW_a$  of  $W_a$  is an embedded curve of  $\mathbb{R}^2 \times \mathbb{P}\mathbb{R}^1$ . The curve  $W_a$  contains  $S_a$  and is not tangent to the weak unstable direction of  $S_a$ , hence  $TW_a$  intersects  $W_{loc}^s(E_a^{uu}; Tf_a)$  at  $T_{S_a}W_a =: (S_a, v_a)$ . We observe that  $a \mapsto v_a \in \mathbb{P}\mathbb{R}^1$  is of class  $C^\infty$ .

As  $(TC_a^n)_a$  contains a segment close to  $W_{loc}^s(E_a^{uu}; Tf_a)$ , there exists a point  $(x_a, y_a) \in \mathbb{R}^2$  so that  $(x_a^n, y_a^n, v_a)_a$  is in  $(TC_a^n)_a$  and is  $C^\infty$ -close to  $(S_a, v_a)_a$ .

Consequently, there exists  $(z_a^n)_a$   $C^\infty$ -small so that  $TC_a^n$  intersects  $TW_a + (z_a^n, 0) = T(W_a + z_a^n)$ . Let  $\rho$  be a compactly supported bump function equals to 1 on  $V$ . We notice that the proposition is proved with the perturbation  $W'_a = W_a + \rho(a) \cdot z_a^n$  which is small for  $n \leq 0$  large.  $\square$



*Remark 30.* Moreover, the tangency point  $\tilde{H}_a^{-n}$  of  $C_a^{-n}$  with  $W'_a$  satisfies that  $(f_a^k(\tilde{H}_a^{-n}))_{k=0}^n$  is close to  $(f_a^{n-k}(H_a))_{k=0}^n$ .

Indeed the curve  $TC_a^{-n}$  being close to be vertical, the point  $\tilde{H}_a^{-n}$  is close to  $f_a^{-n}(H_a)$ . Also the curve  $TC_a^{-n}$  is contracted by  $Tf_a^k$  for every  $a$  for every  $k \leq n$  since  $TC_a^{-n}$  is close to the local stable manifold  $W_{loc}^s(E_a^{uu}; Tf_a)$ .

## 5 Sinks Creation (proof of Prop. 25)

The following section is devoted to prove Proposition 31 below which implies Proposition 25

Let  $V \subset \mathbb{R}^k$  be a compact subset. Let  $(f_a)_{a \in \mathbb{R}^k}$  be a  $C^\infty$ -family of local diffeomorphisms. We suppose that for every  $a \in V$ , the map  $f_a$  has an area contracting saddle point  $P_a$  which displays a persistent homoclinic tangency at  $H_a$  for  $a \in V$ .

In other words, the  $C^\infty$ -family  $(H_a)_a$  formed by the tangency points of the local stable manifolds  $(W_{loc}^s(P_a; f_a))_a$  with a smooth family  $(\Gamma_a)_a$  of embedded segments in  $(W^u(P; f_a))_a$ . Let  $(H_a^{-k})_{k \leq 0}$  be the  $k^{th}$ -preimage of  $H_a$  defined thanks to the inverse branches defining  $\Gamma_a^u$ . We observe that  $H_a^0 = H_a$  and that this presequence converges to  $P_a$ . We define:

$$\mathcal{O}(H_a) = \{H_a^{-k} : k \leq 0\} \cup \{f_a^k(H_a) : k \geq 0\}.$$

**Proposition 31.** *For every  $M \geq 1$  and  $\eta > 0$ , there exists a smooth perturbation  $(f'_a)_a$  such that:*

- for every  $a \notin V$  or  $z \notin B(H_a, \eta)$ , it holds  $f_a(z) = f'_a(z)$ ,
- the map  $f_a$  has a sink  $A_a$  of period at least  $\geq M$  and with orbit in the  $\eta$ -neighborhood of  $\mathcal{O}(H_a)$ .



- the sinks  $A_a$  depends smoothly on  $a \in V$ , and the curve  $(A_a)_{a \in V}$  is  $C^\infty$ -close to  $(H_a)_{a \in V}$ .

*Proof of Proposition 31.* The set  $\mathcal{O}(H_a)$  is discrete and has a unique accumulation point at  $P_a$ . Hence  $\eta = 2 \cdot d(H_a, \mathcal{O}(H_a) \setminus \{\mathcal{O}(H_a)\})$  is positive.

Let  $\lambda_a$  and  $\sigma_a$  be respectively the stable and unstable eigenvalues of  $P_a$ . As  $P_a$  is area contracting it holds:

$$|\lambda_a \sigma_a| < 1.$$

Via a smooth family charts, for every  $a \in V$ , we identify a neighborhood  $N_a$  of  $H_a$  to  $[-1, 1]^2$  so that  $H_a$  is identified to 0, a segment of  $\Gamma_a^u \subset W^u(P_a; f_a)$  to  $[-1, 1] \times \{0\}$  and a segment of  $W^s(P_a; f_a)$  to the graph of  $x \in [-1, 1] \mapsto \rho_a(x)$  with  $\rho_a(0) = 0$  and  $D_0 \rho_a = 0$  for every  $a \in V$ .

We notice that for every  $a \in V$ ,  $H_a^{-k}$  is  $C^\infty$ -close to  $P_a$  for  $k \geq 1$  large.

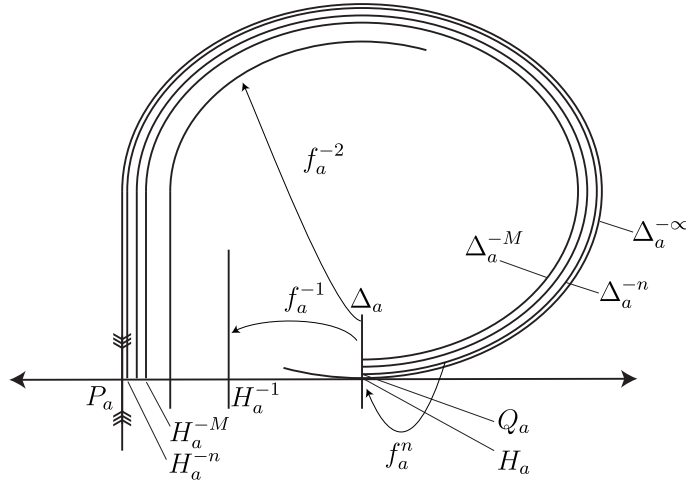


Figure 2: Preimages of  $\Delta_a$ .

Let us first assume that  $H_a$  belongs to the component of  $W^s(P_a; f_a)$  containing  $P_a$ . Let  $\Delta_a^{-\infty}$  be the segment of  $W^s(P_a; f_a)$  with endpoints  $P_a$  and  $H_a$  (by definition they belong to the same component of  $W^s(P_a; f_a)$ ).

Let  $\Delta_a := \{0\} \times [-\delta, \delta]$ , with  $\delta > 0$  small. By the inclination lemma, for  $M$  large enough and  $\delta > 0$  small enough, for every  $n \geq M$ , the preimage of  $\Delta_a$  by  $f_a^n$  contains a segment  $\Delta_a^{-n}$  bounded by  $H_a^{-n}$  and  $\Delta_a$ , which is close to  $\Delta_a^{-\infty}$  (see figure 2). Actually, by the parametric inclination lemma, the family of curves  $(\Delta_a^{-n})_{a \in V}$  is  $C^\infty$ -close to  $(\Delta_a^{-\infty})_{a \in V}$ .

Let us fix  $n \geq M$  large with respect to  $M$ . We observe that  $\Delta_a^{-n} \cap [-1, 1]^2$  is the graph of a function  $x \in [0, 1] \mapsto \rho'_a(x)$ , with  $(\rho'_a)_{a \in V}$   $C^\infty$ -close to  $(\rho_a)_{a \in V}$ .

Let  $Q_a$  be the unique endpoint of  $\Delta_a^{-n}$  in  $\Delta_a$ . We notice that  $(Q_a = (0, \rho'_a(0)))_{a \in V}$  is close to the constant family  $(H_a = 0 = (0, \rho_a(0)))_{a \in V}$ .

Let  $Q'_a \in \Delta_a$  be the image of  $Q_a$  by  $f_a^n$ . Note that  $d(Q_a, Q'_a) < \delta \ll \eta$ .

**Lemma 32.** *For  $n$  large, the family  $(Q'_a)_{a \in V}$  is close to  $(H_a)_{a \in V}$ .*

*Proof.* We observe that  $f_a|_{\Delta_a^{-\infty}}$  is contracting with  $H_a$  as unique fixed point.

By identifying  $\Delta_a^{-k}$  to  $\Delta_a^{-\infty}$  for  $k \geq M$  large (and so  $H_a^{-k}$  to  $P_a$ ), the map  $f_a|_{\Delta_a^{-k}}$  is contracting with  $H_a$  as unique fixed point.

As  $(\Delta_a^{-k})_{a \in V}$  is  $C^\infty$ -close to  $(\Delta_a^{-\infty})_{a \in V}$  we have uniform bounds which enable us to prove that  $(f_a^{n-M}(Q_a))_{a \in V}$  is  $C^\infty$ -close to  $(H_a^{-M})_{a \in V}$  for  $n$  large.

By taking  $n$  large in function of  $M$ , it comes that  $(Q'_a)_{a \in V} = (f_a^n(Q_a))_{a \in V}$  is  $C^\infty$ -close to  $(H_a)_{a \in V}$ .  $\square$

**Lemma 33.** *The  $n$ -first iterates of  $Q_a$  are in the  $\eta$ -neighborhood of  $\mathcal{O}(H_a)$ .*

*Proof.* By taking the notation of the previous proof, since  $\delta > 0$  is small depending on  $\eta$ , the  $M$  first iterates of  $Q_a$  are  $\eta$ -close to the  $M$  first iterates of  $H_a$ . In the previous proof we saw that  $\{f_a^k(Q_a) : n \geq k \geq M\}$  is even closer to  $\{H_a^{-k} : 0 \leq k \leq n - M\}$ .  $\square$

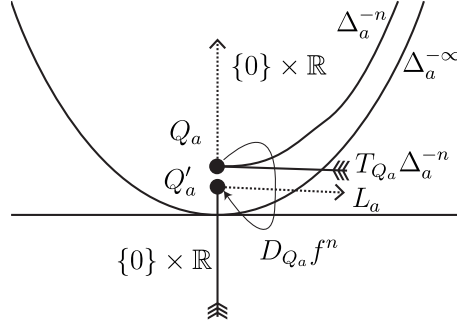


Figure 3: Notations involved.

We recall that  $f_a^n(Q_a) = Q'_a$  and  $Df_a^n(T_{Q_a}\Delta_a^{-n}) = Q'_a + \{0\} \times \mathbb{R}$ . Let  $L_a \in \mathbb{P}\mathbb{R}^1$  be defined by  $Q'_a + L_a := Df_a^n(Q_a + \{0\} \times \mathbb{R})$ . By the inclination lemma, the lines family  $(L_a)_{a \in V}$  is  $C^\infty$ -close to  $(\mathbb{R} \times \{0\})_{a \in V}$ . As  $(Q_a)_{a \in V}$  is close to  $(0)_{a \in V}$  and  $(\Delta_a^{-n})_{a \in V}$  is close to  $(\Delta_a^{-\infty})_{a \in V}$ , the family  $(T_{Q_a}\Delta_a^{-n})_{a \in V}$  is close to  $(T_{H_a}\Delta_a^{-\infty})_a = (\mathbb{R} \times \{0\})_a$  and so to  $(L_a)_{a \in V}$ .

Consequently, there exists a family  $(f'_a)_{a \in V}$  which is  $C^\infty$ -close to  $(f_a)_{a \in V}$  so that:

- $f'_a(Q'_a) = f_a(Q_a)$  for every  $a \in V$ ,
- $Df'_a(Q'_a + L_a) = Df_a(T_{Q_a}\Delta_a^{-n})$  for every  $a \in V$ ,
- $Df'_a(Q'_a + \{0\} \times \mathbb{R}) = Df_a(Q_a + \{0\} \times \mathbb{R})$  for every  $a \in V$ .
- $f'_a(z) = f_a(z)$  if  $a \notin V$  or  $z \notin B(H_a, \eta)$ .

By Lemma 33, the  $n$ -first  $f_a$ -iterates of  $Q_a$  are included in the  $\eta$  neighborhood of  $\mathcal{O}(H_a)$ . Thus  $\{f_a^k(Q_a) : 1 \leq k \leq n - 1\}$  are  $\eta$ -distant to  $H_a$ . Thus it holds for every  $a \in V$ :

$$D_{f'_a(Q'_a)} f_a^{n-1} = D_{f_a(Q_a)} f_a^{n-1}$$

Consequently the point  $Q'_a$  is  $n$ -periodic and  $D_{Q'_a} f_a^n$  sends  $L_a$  to  $\{0\} \times \mathbb{R}$  with a contraction factor of the order of  $\lambda_a^n$  and it sends  $\{0\} \times \mathbb{R}$  to  $L_a$  with an expansion factor of the order of  $\sigma_a^n$ . As  $\lambda_a \sigma_a < 1$ ,  $D_{Q'_a} f_a^{2n}$  is a contracting homotety of factor of the order of  $(\lambda_a \sigma_a)^{2n}$ . In particular  $Q'_a$  is a sink of period  $n \geq M$ .

If  $H_a$  does not belong to the component of  $W^s(P_a; f_a)$  containing  $P_a$ , then let  $k \geq 0$  be minimal such that  $f_a^k(H_a)$  belongs to the connected component of  $W^s(P_a; f_a)$ . By  $f_a$ -invariance of  $W^u(P_a; f_a)$ , this immersed submanifold is still tangent to  $W^s(P_a; f_a)$  at  $f_a^k(H_a)$ . Hence by the above argument, for small perturbation of  $W^u(P_a; f_a)$  around  $f_a^k(H_a)$  there is a persistent homoclinic tangency. The map  $f_a^k$  being a diffeomorphism on a neighborhood of  $H_a$ , we can make this perturbation around  $H_a$ .  $\square$

## 6 Proof that Proposition 27 implies main Theorem A

Let us assume Proposition 27, there is a dense set in  $\mathcal{U}^{d,\infty}$  formed by families  $(f_a)_a \in \mathcal{U}^\infty$  which satisfies the following property:

There exists a finite open covering  $(U_i)_i$  of  $\mathbb{I}^k$  so that for every  $i$  there exist a periodic, projectively hyperbolic source  $(S_{ia})_{a \in U_i}$  and a continuous family of embedded segments  $(\Gamma_{ia}^u)_a$  of  $(W^u(P_a; f_a))_a$  such that:

- (i)  $S_{ia}$  is in  $\Gamma_{ia}^u \subset W^u(P_a; f_a)$ , for every  $a \in U_i$ .
- (ii)  $T_{S_{ia}} \Gamma_{ia}^u$  is not the weak unstable direction of  $S_{ia}$ , for every  $a \in U_i$ .
- (iii) There exists a (smooth) curve of points  $(H_{ia})_{a \in U_i}$  in  $(W^s(P_a; f_a))_{a \in U_i}$  at which  $W^s(P_a; f_a)$  has a quadratic tangency with the strong unstable foliation of  $S_{ia}$ , for every  $a \in U_i$ .
- (iv) For every  $a \in U_i \cap U_j$  with  $i \neq j$ , the sets  $(f_a^k(H_{ia}))_{k \geq 0}$  and  $(f_a^k(H_{ja}))_{k \geq 0}$  are disjoint and the orbits  $(f_a^k(S_{ia}))_{k \geq 0}$  and  $(f_a^k(S_{ja}))_{k \geq 0}$  are disjoint.

We want to show that under these assumptions for every  $M > 0$ , each of these families can be perturbed – following the algorithm described in the sketch of proof – to one which displays a sink of period at least  $M$  for every  $a \in \mathbb{I}^k$ .

In order to do so, we need to handle independently different perturbations. Hence we need to find some independent rooms in the phase space to do so.

For every  $i$  and  $a \in U_i$ , we define the following sets of forward and backward orbits.

Put  $\mathcal{O}^+(H_{ia}) := \{f_a^k(H_{ia}) : k \geq 0\}$  and  $\mathcal{O}^+(S_{ia}) := \{f_a^k(S_{ia}) : k \geq 0\}$ .

Let  $\mathcal{O}^-(H_{ia})$  be the preorbit of  $H_{ia}$  defined thanks to the inverse branches defining the strong unstable foliation of  $S_{ia}$ . We notice that  $\mathcal{O}^-(H_{ia})$  accumulates on  $\mathcal{O}^+(S_{ia})$ . We put:

$$\mathcal{O}(H_{ia}) := \mathcal{O}^+(H_{ia}) \cup \mathcal{O}^-(H_{ia}).$$

Also the segment  $\Gamma_{ia}^u$  of  $W^u(P_a; f_a)$  is defined by a sequence of inverse branches of  $f_a$ . Let  $\mathcal{O}^-(S_{ia}) := (S_{ia}^{(k)})_{k \leq -1}$  be the preorbit of  $S_{ia}$  associated to this sequence of inverse branches. We notice that  $\mathcal{O}^-(S_{ia})$  is discrete with  $P_a$  as unique accumulation point. We put:

$$\mathcal{O}(S_{ia}) := \mathcal{O}^+(S_{ia}) \cup \mathcal{O}^-(S_{ia}).$$

**Fact 34.** *We can assume that  $\mathcal{O}^+(S_{ia})$  is disjoint from  $\mathcal{O}^-(S_{ia})$ .*

*Proof.* Indeed,  $\mathcal{O}^+(S_{ia})$  is finite since  $S_{ia}$  is periodic thus  $\mathcal{O}^-(S_{ia})$  is not contained in  $\mathcal{O}^+(S_{ia})$ . If the preimage  $S_{ia}^{(-1)}$  of  $S_{ia}$  is in  $\mathcal{O}^+(S_{ia})$ , then properties (i) – ii – iii – iv) are still satisfied by  $S_{ia}^{(-1)}$  and the preimage  $\Gamma_{ia}'^u$  of  $\Gamma_{ia}^u$ . Consequently, by shifting the orbit, we can assume that  $S_{ia}^{(-1)}$  is not in  $\mathcal{O}^+(S_{ia})$ , and so  $\mathcal{O}^+(S_{ia})$  is disjoint from  $\mathcal{O}^-(S_{ia})$ .  $\square$

By (iv), for every  $i \neq j$  such that  $a \in U_i \cap U_j$ , the sets  $\mathcal{O}^+(S_{ia})$  and  $\mathcal{O}^+(S_{ja})$  are disjoint. This implies that the sets  $\mathcal{O}(S_{ia})$  and  $\mathcal{O}(S_{ja})$  are disjoint.

We notice also that  $\mathcal{O}(S_{ia})$  is disjoint from  $\mathcal{O}(H_{ja})$ , for every  $i, j$  such that  $a \in U_i \cap U_j$  (otherwise the sink would converge to  $P_a$ ). Consequently:

**Fact 35.** *For every  $a$  and all  $i \neq j$  such that  $a \in U_i \cap U_j$ , the point  $S_{ia}^{(-1)}$  is disjoint from the set  $\mathcal{O}(S_{ja}) \cup \mathcal{O}(H_{ja}) \cup \mathcal{O}(H_{ia}) \cup \{P_a\}$ .*

Note also that  $\mathcal{O}(S_{ia})$  is discrete with a unique accumulation point at  $P_a$  and  $\mathcal{O}(H_{ia})$  is discrete with accumulation points  $P_a$  and the finite set  $\mathcal{O}^+(S_{ia})$ . Hence the set  $\mathcal{O}(S_{ja}) \cup \mathcal{O}(S_{ia}) \cup \mathcal{O}(H_{ja}) \cup \mathcal{O}(H_{ia}) \cup \{P_a\}$  is compact and  $S_{ia}^{(-1)}$  is isolated therein.

As these compact sets depend continuously on  $a$ , by shrinking slightly the covering  $(U_i)_i$ , we obtain:

**Lemma 36.** *There exists  $\delta > 0$  so that for every  $i \neq j$  and  $a \cap U_i \cap U_j$ , the point  $S_{ia}^{(-1)}$  is at distance at least  $3\delta > 0$  to  $\mathcal{O}(S_{ja}) \cup (\mathcal{O}(S_{ia}) \setminus \{S_{ia}^{(-1)}\}) \cup \mathcal{O}(H_{ja}) \cup \mathcal{O}(H_{ia})$ .*

Let  $(U'_i)_i$  be an open, relatively compact covering of  $\mathbb{I}^k$  such that  $cl(U'_i) \subset U_i, \forall i$ .

By shrinking  $\Gamma_{ia}^u$ , we can assume its preimage  $\Gamma_{ia}^{u(-1)}$  included in the  $\delta/2$ -neighborhood of  $S_{ia}^{(-1)}$  for every  $i$  and  $a \in U_i$ . By Proposition 29, we can perturb  $\Gamma_{ia}^u$  to a curve  $\tilde{\Gamma}_{ia}^u$  which is tangent to  $W^s(P_a; f_a)$  at a point  $\tilde{H}_{ia}$ .

By Remark 30, the forward orbit of  $\mathcal{O}^+(\tilde{H}_{ia}) := \{f_a^k(\tilde{H}_{ia}) : k \geq 0\}$  is included in the  $\delta$ -neighborhood of  $\mathcal{O}(H_{ia})$ .

As  $S_{ia}^{(-1)}$  is  $3\delta$ -distant from  $\mathcal{O}(H_{ia}) \cup (\mathcal{O}(S_{ia}) \setminus \{S_{ia}^{(-1)}\})$  we can handle a perturbation  $(f_{ia})_a$  of  $(f_a)_a$  which is supported by  $a \in U_i$  or  $z \in B(S_{ia}^{(-1)}, \delta)$  so that for every  $a \in U'_i$ , the curve  $\tilde{\Gamma}_{ia}^u$  is the hyperbolic continuity of  $\Gamma_{ia}^u$  for  $f_{ia}$ . This proves:

**Lemma 37.** *For every  $i$  there exists a smooth perturbation  $(f_{ia})_{a \in U_i}$  of  $(f_a)_{a \in U_i}$  so that:*

- the hyperbolic continuity of  $\tilde{\Gamma}_{ia}^u$  of  $\Gamma_{ia}^u$  has a persistent homoclinic quadratic tangency with  $W^s(P_a; f_{ia})$  at a point  $\tilde{H}_{ia} \in \tilde{\Gamma}_{ia}^u \cap B(S_{ia}, \delta)$  and with forward orbit  $\mathcal{O}^+(\tilde{H}_{ia})$  in the  $\delta$ -neighborhood of  $\mathcal{O}(H_{ia})$ , for every  $a \in U'_i$ .
- $f_{ia}(z) = f_a(z)$  for every  $z \notin B(S_{ia}^{(-1)}, \delta)$  and  $a \in U_i$ .

We define  $\mathcal{O}^-(\tilde{H}_{ia})$  by different branches from those defining  $\mathcal{O}^-(H_{ia})$ : we consider the inverse branches of the dynamics defining  $\Gamma_{ia}^u$ . This defines a sequence of preimages  $(\tilde{H}_{ia}^{(k)})_{k \leq 0}$  so that  $\tilde{H}_{ia}^{(k)}$  is close to  $S_{ia}^{(k)}$  for every  $k \leq -1$ .

We put  $\mathcal{O}(\tilde{H}_{ia}) := \mathcal{O}^-(\tilde{H}_{ia}) \cup \mathcal{O}^+(\tilde{H}_{ia})$ . It is a discrete set with a unique accumulation point  $P_a$ . Also we notice that for every  $i \neq j$  and  $a \in U'_i \cap U'_j$ :

- the  $\delta$ -neighborhood of  $\mathcal{O}(\tilde{H}_{ja})$  is disjoint to  $B(S_{ia}^{(-1)}, \delta)$ ,
- $B(S_{ia}^{(-1)}, \delta)$  contains  $\tilde{H}_{ia}^{(-1)}$  and is disjoint to the  $\delta$ -neighborhood of  $\mathcal{O}(\tilde{H}_{ia}) \setminus \{\tilde{H}_{ia}^{(-1)}\}$ .

Then Proposition 31 implies:

**Lemma 38.** *For every  $M \geq 0$ , for every  $i$  there exists a smooth perturbation  $(f_{ia})_{a \in U_i}$  of  $(f_a)_{a \in U_i}$  so that:*

- $(f_{ia})_a$  has a sinks  $A_{ia}$  of period  $\geq M$  with orbit in  $B(\mathcal{O}(H_{ia}), \delta)$ , for every  $a \in U'_i$ .
- $f_{ia}(z) = f_a(z)$  for every  $z \notin B(S_{ia}^{(-1)}, \delta)$  and  $a \in U_i$ .

Note that by the first item, for every  $a \in U_i \cap U'_j$  with  $i \neq j$ , the orbit of  $A_{ja}$  does not intersect  $B(S_{ia}^{(-1)}, \delta)$ . Hence for every perturbation  $(f'_a)_a$  of  $(f_a)_a$  so that:

- $f'_a(z) = f_a(z)$  for every  $z \notin B(S_{ia}^{(-1)}, \delta)$  and every  $i$  such that  $a \in U_i$ ,
- $f'_a = f_{ia}(z)$  for every  $a \in U'_i$  and every  $z \in B(S_{ia}^{(-1)}, \delta)$ ,

The point  $A_{ia}$  is still an attracting cycle of period  $\geq M$  for  $f'_a$ , for every  $a \in U'_i$ . As  $(U'_i)_i$  is a covering of  $\mathbb{I}^k$ , the family  $(f'_a)_a$  displays a sink of period  $\geq M$  for every  $a \in \mathbb{I}^k$ .

We notice that the perturbation  $(f'_a)_a$  exists since the sets  $(\{(a, z) \in U_i \times M : z \in B(S_{ia}^{(-1)}, \delta)\})$  are disjoint.

## 7 Replications of the source $S$ (Proof of Prop. 27)

Let  $(f_a)_a$  be a smooth family in  $\mathcal{U}^\infty \subset \mathcal{U}^{d,\infty}$ . Let  $\mathcal{F}_a^{uu}(S_a)$  be the strong unstable foliation associated to the projectively hyperbolic fixed source  $S_a$  of  $f_a$ .

**Proposition 39.** *For every  $N$  and every  $\epsilon > 0$ , for every  $a_0$ , there exist  $(k+1)3^k$ -sources  $(S_{ia_0})_i$  with disjoint periodic orbits so that:*

- (1) *the source  $S_{ia_0}$  is projectively hyperbolic and is  $\epsilon$ -close to  $S_{a_0}$  and the  $C^d$ -jet  $J_{a_0}^d(S_{ia})_a$  of  $(S_{ia})_a$  at  $a = a_0$  is  $\epsilon$ -close to  $J_{a_0}^d(S_a)_a$ .*
- (2) *the set  $B$  is included in the basin of  $S_{ia_0}$  and the leaves of the strong unstable foliation associated to  $\mathcal{F}^{uu}(S_{ia_0})$  are  $\epsilon$ - $C^2$ -close to those of  $\mathcal{F}^{uu}(S_{ia_0})$ , over  $B$ .*

*Proof.* For the sake of simplicity, we prove this proposition for:

$$a_0 = 0.$$

Let  $\underline{Q} \in \overleftarrow{K}$  be such that  $S_0$  is in  $W_{loc}^u(\underline{Q}; f_0)$ . The preorbit  $\underline{Q}$  being not necessarily periodic, it does not need to have Lyapunov exponents well defined. Let  $(\lambda_n)_n$  and  $(\sigma_n)_n$  be defined by:

$$\lambda_n = \|D_Q f_0^{-n} |E^s|\|^{-1} < 1 \quad \text{and} \quad \sigma_n = \|D_Q f_0^{-n} |E^u|\|^{-1} > 1.$$

Let  $\sigma_u$  and  $\sigma_{uu}$  be the weak and strong unstable eigenvalues of  $S_0$ .

Let  $\phi_a$  be the inverse branch of  $f_a$  which contracts  $B$  to  $S_a$ . Then  $\phi_0^n$  contracts  $B$  to a small neighborhood  $B_n$  of  $S_0$ , with a contraction of the order of  $\sigma_u^{-n}$ . Let  $(\psi_a^m)_{m \geq 0}$  be the inverse branches of  $(f_a^m)_{m \geq 0}$  which define  $W^u(\underline{Q}; f_a)$ . For every  $m$  large,  $\psi_0^m$  is defined from a small neighborhood of  $S_0$  with image in a small neighborhood of the zero coordinate of  $\overleftarrow{f}^{-m}(\underline{Q})$ . Its Lipschitz constant is of the order of  $\lambda_m^{-1}$ . The following lemma is obvious:

**Lemma 40.** *For every  $m$  large, for every  $n \geq 0$  large enough, it holds:*

- (a)  $\lambda_m \cdot \sigma_u^n$  is large,
- (b)  $\lambda_m \cdot \sigma_u^n$  is small with respect to  $\sigma_m \cdot \sigma_{uu}^n$ .

In particular, for  $n$  large enough, for such a choice of  $n, m$ , the map  $\phi_0^n \circ \psi_0^m \circ \phi_0^n$  is well defined on  $B$  and is very contracting. We put  $B_{n,m} = \psi_0^m(B_n) = \psi_0^m \circ \phi_0^n(B)$  and  $B_{n,m,n} = \phi_0^n(B_{n,m}) = \phi_0^n \circ \psi_0^m \circ \phi_0^n(B)$ .

Also  $B_{n,m,n}$  is included in the small neighborhood  $B_n \subset B$  of  $S_0$ , and so in  $B$ . Consequently  $\phi_0^n \circ \psi_0^m \circ \phi_0^n$  has a fixed point  $S_{i_0}$  in  $B_{n,m,n}$ . As  $B_n$  is a small neighborhood of  $S_0$ , the  $f_0$ -periodic source  $S_{i_0}$  is close to  $S_0$ . We notice that  $B$  is in the basin of  $S_{i_0}$ . Let us show that the continuation  $(S_{ia})_a$  of  $S_{i_0}$  satisfies (1) – (2) for  $n, m$  sufficiently large.

**(1)** Let us come back to the notations introduced in §2.2 on the space of  $d$ -jets  $J_0^d(\mathbb{I}^k, M)$ . We remark that the map  $J_0^d(\phi_a^n)_a$  is as contracting as  $\phi_0^n$  with a constant of the order of  $\sigma_u^{-n}$ , whereas  $J_0^d(\psi_a^m)_a$  is Lipschitz with a constant of the order of  $\lambda_m^{-1}$ . Hence, by (a), the composed map  $J_0^d(\psi_a^n \circ \phi_a^m \circ \phi_a^n)_a$  is very contracting with a unique fixed point  $J_0^d(S_{ia})_a$  close to  $J_0^d(S_a)_a$ .

By hyperbolicity and condition (iv), every line  $L$  close to the line field  $E^{uu}(S_0)$  and pointed at a point in  $B_{n,m,n}$ , is sent by  $Df^m$  to a line  $L'$  even closer to the line field  $E^{uu}(S_0)$  and pointed at a point in  $B_{n,m}$ . Then it is sent by  $Df^m$  to a line  $L''$  close to  $TW^u(\underline{Q}; f_0)$  (and pointed at a point in  $B_n$ ). Finally, it is sent by  $Df^n$  to a line very close to  $E^{uu}(S_0)$  and pointed at a point in  $B$ . Hence there is an invariant cone field, and so  $S_{i_0}$  is a projectively hyperbolic periodic source.

(2) We proved that the strong unstable direction  $E_i^{uu}(S_0)$  associated to  $S_{i0}$  is  $C^0$ -close to  $E^{uu}(S_0)$ . Let us show that the leaves integrating  $E_i^{uu}(S_0)$  are  $C^2$ -close to the leaves of  $\mathcal{F}^{uu}(S_0)$ . Let  $\gamma$  be a curve in  $B_{n,m,n}$  which is  $C^2$ -close to a plaque of  $\mathcal{F}^{uu}(S_0)$ . By projective hyperbolicity, the image by  $f_0^n$  of  $\gamma$  is a curve  $C^2$ -close to a plaque of  $\mathcal{F}^{uu}(S_0)$  in  $B_{n,m}$ . Then by the inclination lemma, its image by  $f_0^m$  is  $C^2$ -close to a segment of  $W_{loc}^u(Q; f_0)$  in  $B_m$ . Again by projective hyperbolicity, its image by  $f_0^n$  is then  $C^2$ -close to a plaque of  $\mathcal{F}^{uu}(S_0)$ . This shows that a curve in  $B_{n,m,m}$  which is  $C^2$ -close to a plaque in  $\mathcal{F}^{uu}(S_0)$  is sent by  $f_0^{2n+m}$  to a curve which is even closer to a plaque of  $\mathcal{F}^{uu}(S_0)$ . This proves that the leaves  $\mathcal{F}_i^{uu}(S_0)$  are  $C^2$ -close to those of  $\mathcal{F}^{uu}(S_0)$ .

Consequently for every  $m$  large, for every  $n$  large depending on  $m$ , there is a  $2n+m$ -periodic sources  $(S_{ia})_a$  which satisfies (1–2) at  $a = 0$ . By taking different values for  $m$ , we get  $(k+1)3^k$  periodic sources  $(S_{ia})_a$  with disjoint orbits as claimed in the Lemma.  $\square$

For every  $a \in \mathbb{I}^k$ , all the estimates involved in the algorithm given in the above lemma are bounded from above. Hence by compactity of  $\mathbb{I}^k$ , the integers  $n, m$  are bounded by a constant  $N$  independent of  $a \in \mathbb{I}^k$ .

This implies the following:

**Corollary 41.** *There exist  $\eta > 0$  and  $N > 0$  so that for every  $a_0 \in \mathbb{I}^k$ , there exists a family  $(S_{ia_0})_{i \in \hat{J}(a_0)}$  of periodic sources so that:*

- (a) *The cardinality of  $\hat{J}(a_0)$  is  $(k+1)3^k$  and for every  $i \neq j \in \hat{J}(a_0)$ , the sources  $S_{ia_0}$  and  $S_{ja_0}$  have disjoint orbits,*
- (b) *For every  $i \in \hat{J}(a_0)$ , the source  $S_{ia_0}$  persists for every  $a_1 \in a_0 + [-\eta, \eta]^k$ , and its continuations  $(S_{ia})_a$  satisfy (1) & (2) at  $a_1$ .*
- (c) *The period of  $S_{ia_0}$  is bounded by  $3N$  and the expansion of  $S_{ia}$  is bounded from below by 1000 for every  $a \in a_0 + [-\eta, \eta]^k$ .*

For every  $a_0 \in \mathbb{I}^k$ , for every  $i \in \hat{J}(a_0)$ , for every  $a \in a_0 + [-\eta, \eta]^k$ , we define the finite set:

$$\mathcal{O}^+(S_{ia}) := \{f_a^k(S_{ia}) : k \geq 0\}.$$

By (c), there exists  $\delta > 0$ , s.t. for all  $a_0 \in \mathbb{I}^k$  and  $i \in \hat{J}(a_0)$ , the map  $f_{a_0}^{(3N)!}|_{B(S_{ia_0}, \delta)}$  is expanding and so  $S_{ia_0}$  is its unique fixed point. Hence, for all  $a_0, a_1 \in \mathbb{I}^k$ , so that  $a_0 + [-\eta, \eta]^k \cap a_1 + [-\eta, \eta]^k$  contains a parameter  $a$ , for all  $i \in \hat{J}(a_0)$  and  $j \in \hat{J}(a_1)$ , it holds:

$$\text{either } \mathcal{O}^+(S_{ia}) = \mathcal{O}^+(S_{ja}) \quad \text{or either} \quad d(\mathcal{O}^+(S_{ia}), \mathcal{O}^+(S_{ja})) > \delta.$$

For every  $\eta' < \eta$ , let  $\mathbb{Z}_{\eta'} := \mathbb{I}^k \cap \eta' \mathbb{Z}^k$ .

**Lemma 42.** *For every  $\eta' < \eta$ , there exists  $\prod_{a_0 \in \mathbb{Z}_{\eta'}} J(a_0) \subset \prod_{a_0 \in \mathbb{Z}_{\eta'}} \hat{J}(a_0)$  so that:*

- *for all  $a_0 \in \mathbb{Z}_{\eta'}$ , the set  $J(a_0)$  has cardinality  $(k+1)$ ,*
- *for all  $(a_0, i) \neq (a_1, j) \in \prod_{a_0 \in \mathbb{Z}_{\eta'}} J(a_0)$ , for every  $a \in a_1 + [-\eta', \eta']^k \cap a_0 + [-\eta', \eta']^k$ , the orbits  $\mathcal{O}(S_{ia})$  and  $\mathcal{O}(S_{ja})$  are  $\delta$ -distant.*

*Proof.* Let us index  $\mathbb{Z}_{\eta'} =: \{a_i : 1 \leq i \leq q\}$ . Let  $J(a_1) \subset \hat{J}(a_1)$  be any subset of cardinality  $k+1$ .

Let  $2 \leq q' \leq q$  and assume by induction  $J(a_i)$  constructed for every  $i < q'$ . We notice that the cardinality of  $\{a_i \in a_{q'} + [-\eta', \eta']^k : i < q'\}$  is at most  $3^k - 1$ . Hence we have to remove at

most  $(k+1)(3^k-1)$  periodic sources of  $\hat{J}(a_{q'})$  so that the remaining sources have continuations with disjoint orbit to those indexed by  $\cup_{\{a_i \in a_{q'} + (-\eta', \eta')^k : i < q'\}} J(a_i)$ . We chose any set  $J(a_{q'})$  of cardinality  $k+1$  in the remaining set formed by at least  $(k+1)3^k - (k+1)(3^k-1) = (k+1)$  different sources.  $\square$

We recall that there exists a continuous family of local unstable manifolds  $(W_{loc}^u(z; f))_{z \in \overleftarrow{K}}$  so that by  $(H_3)$  and Proposition 39.2, for every  $(a_0, i) \in \prod_{a_0 \in \mathbb{Z}_{\eta'}} J(a_0)$ , there exists  $\underline{z}_i \in \overleftarrow{K}$  satisfying:

- $W_{loc}^u(\underline{z}_i; f_{a_0})$  contains  $S_{i, a_0}$ ,
- there exists a  $C^d$ -curve of points  $(Q_a)_a \in (W_{loc}^u(\underline{z}_i; f_a))_a$  and a continuous function  $\epsilon_i$  equal to zero at 0 s.t:

$$d(Q_a, S_{i, a}) \leq \epsilon_i(\|a - a_0\|) \cdot \|a - a_0\|^d.$$

We recall that the family  $(W_{loc}^u(z; f_a)_{a \in \mathbb{I}^k})_{z \in \overleftarrow{K}}$  is continuous for the  $C^\infty$  topology.

Hence there exists a family of smooth charts  $(\phi_{ia})_{a \in \mathbb{I}^k}$  from a neighborhood of  $W_{loc}^u(\underline{z}_i; f_a)$  onto an open subset of  $\mathbb{R}^2$ , which send  $W_{loc}^u(\underline{z}_i; f_a)$  to the constant segment  $[-1, 1] \times \{0\}$  for every  $a \in \mathbb{I}^k$  and which have bounded  $C^{r, r}$ -norm independently of  $(a_0, i)$  for every  $r \geq 0$ .

We remark that  $S_{ia}$  belongs to the domain of this chart for  $a$  sufficiently close to  $a_0$ . Let  $(x_i(a), y_i(a)) := \phi_a(S_{ia})$  and remark that  $\partial_a^s y_i(a_0) = 0$  for every  $s \leq d$ .

Moreover, by Corollary 41.(c), the  $C^{d+1}$ -norm of  $(S_{i, a})_{a \in a_0 + [-\eta, \eta]^k}$  is bounded independently of  $(a_0, i)$ . Thus there exists  $C_{d+1} > 0$  so that for every  $a \leq \eta$ , it holds:

$$|\partial_a^{d+1} y_i(a_0 + a)| \leq C_{d+1}.$$

A crucial point is that  $C_{d+1}$  and  $\delta > 0$  do not depend on  $\eta'$  nor on  $(a_0, i)$ .

Hence for  $\eta' \in (0, \eta)$  sufficiently small, we can  $C^{d, \infty}$ -perturb  $(f_a)_a$  to a smooth family  $(f'_a)_a$  so that  $S_{ia}$  persists as  $S'_{ia} := \phi_a^{-1}(x_{ia}, 0)$  for every  $a \in a_0 + [-2\eta'/3, 2\eta'/3]$ , the perturbation being supported by  $(a, z) \in a_0 + [-\eta', \eta']^k \times B(S_{ia}, \delta/2)$ .

Note that the perturbation is  $C^{d, \infty}$ -small when  $\eta'$  is small. This proves:

**Proposition 43.** *For every  $\eta' < \eta$  small, there exists a  $C^{d, \infty}$ -perturbation  $(f'_a)_a \in \mathcal{U}^\infty$  of  $(f_a)_a$  and families of sources  $(S'_{ia_0})_{a_0 \in \mathbb{Z}_{\eta'}, i \in J(a_0)}$  so that*

- for all  $a_0 \in \mathbb{Z}_{\eta'}$ , the cardinal of the set  $J(a_0)$  is  $(k+1)$ ,
- for all  $(a_0, i) \neq (a_1, j) \in \prod_{a_0 \in \mathbb{Z}_{\eta'}} J(a_0)$ , for every  $a \in a_1 + [-\eta', \eta']^k \cap a_0 + [-\eta', \eta']^k$ , the orbits  $\mathcal{O}(S'_{ia})$  and  $\mathcal{O}(S'_{ja})$  are  $\delta/2$ -distant.
- for all  $a_0 \in \mathbb{Z}_{\eta'}$ , for all  $i \in J(a_0)$ , there exists  $\underline{z}_i \in \overleftarrow{K}$  so that  $S'_{ia}$  belongs to  $W_{loc}^u(\underline{z}_i; f_a)$  for every  $a \in a_0 + [-2\eta'/3, 2\eta'/3]^k$ .

By Taking  $\eta'$  such that  $1/\eta' \in \mathbb{N}$ , it holds that  $\cup_{a_0 \in \mathbb{Z}_{\eta'}} a_0 + [-2\eta'/3, 2\eta'/3]^k \supset \mathbb{I}^k$ .

Note that by  $(H_2)$  a segment  $W_{loc}^s(P, f'_a)$  has a robust tangency with  $\mathcal{F}^{uu}(S_a)$  for every  $a \in \mathbb{I}^k$ . Also if  $a \in a_0 + (-\eta, \eta)^k$  and  $a_0 \in \mathbb{Z}_{\eta'}$ , and  $i \in J(a_0)$ , the leaves of the foliation  $\mathcal{F}^{uu}(S_{ia})$  are  $C^2$ -close to  $\mathcal{F}^{uu}(S_a)$ . We recall that  $\mathcal{F}^{uu}(S_{ia})$  is actually a  $C^\infty$ -foliation depending smoothly on the parameter by Prop. 28.

Since  $\mathcal{U}$  is a trace of a  $C^1$ -open set, this does not imply that the tangency is quadratic. However, by performing a small perturbation of the dynamics in the  $\delta/2$ -neighborhood of  $S'_{ia}$ ,

we can keep  $S'_{ia}$  and  $W^s_{loc}(P, f'_a)$  at the same places and make any small smooth perturbation for  $(\mathcal{F}^{uu}(S_{ia}))_a$ .

By Thom's transversality theorem, for a typical perturbation of  $(\mathcal{F}^{uu}(S_{ia}))_a$ , the set of parameters for which  $W^s_{loc}(P, f'_a)$  does not display a quadratic tangency with  $\mathcal{F}^{uu}(S_{ia})$  is a (possibly empty) submanifold of codimension 1. Let  $U_i$  be the complement of this manifold in  $(-2\eta/3, 2\eta/3) + a_i$ .

Moreover these  $k + 1$  one-co-dimensional manifolds are multi-transverse for  $i \in J(a_0)$ . As the parameter space  $\mathbb{I}^k$  has dimension  $k$ , for a typical perturbation of  $((\mathcal{F}^{uu}(S_{ia}))_a)_{i \in J(a_0)}$ , the union  $\cup_{i \in J(a_0)} U_i$  contains  $a_0 + (-2\eta'/3, 2\eta'/3)$ .

Thus a  $C^{d,\infty}$ -perturbation  $(f''_a)_a \in U^\infty$  of  $(f_a)_a$  satisfies Proposition 27 with sources  $(S_{ia})_{a \in U_i}$  among  $i \in \cup_{a_0 \in \mathbb{Z}_{\eta'}} J(a_0)$ .

## 8 Proof of Corollary B

*Proof.* Let  $X \in \{A, PS\}$  and  $1 \leq d \leq r < \infty$ . Let us come back to Example 20. Let  $N$  be a small neighborhood of  $(D \sqcup I_S \sqcup I_P \sqcup I_{P'}) \times [-3, 3]$ .

We recall that for every  $M \geq 0$ , we have constructed a  $\mathcal{U}_X^{d,r}$ -dense set  $D$  of  $C^{d,\infty}$ -families  $(f_a)_a$  so that for every  $a \in \mathbb{I}^k$ , the map  $f_a$  displays a sink of period at least  $M$  with orbit in  $N$ .

Let  $\hat{\mathbb{R}}$  be the one point compactification of  $\mathbb{R}$ , and let  $\mathbb{A} = \hat{\mathbb{R}} \times [-4, 4]$ . Let  $I$  be a compact neighborhood of the infinity, so that  $I \times [-4, 4]$  does not intersect  $N$ .

As the map  $Q: \sqcup_{\delta \in \Delta} I_\delta \sqcup I_S \sqcup I_P \rightarrow [-1, 1]$  is orientation preserving and  $f_a|_{I_{P'} \times [-3, 3]}$  as well, it is possible to extend each smooth family  $(f_a)_a \in D$  to a family on local diffeomorphisms of  $\mathbb{A}$  of degree  $\text{Card } \Delta + 3$  so that:

- (i) for every  $(f_a)_a \in D$ , for every  $a \in \mathbb{I}^k$ , the map has a sinks of period  $\geq M$  outside of  $I \times [-4, 4]$ .
- (ii) for every  $\theta \in \hat{\mathbb{R}}$ , the map  $f|_{\{\theta\} \times [-4, 4]}$  is injective.

By the algorithm written section 5 [Ber16b], this implies Theorem B. Let us recall briefly the flavor of the argument.

For every  $n \geq 3$ , we construct a compact bundle inside a unit  $n$ -ball with fibers of dimension  $n - 2$ , and a family of dynamics  $(\hat{f}_a)_a$  which preserves this fibration. Moreover, we assume that the fibers are very contracted, this implies that this fibration is of class  $C_A^{d+r, d+r}$  and persistent [Berb]. Also we assume that the base dynamics is equal to  $(f_a|_N)_a$  with  $(f_a)_a \in D$ . These dynamics are shown to be extendable to any  $n$ -manifold.

Then every perturbation  $(\hat{f}'_a)_a$  of  $(\hat{f}_a)_a$  fiber over a  $C_A^{d+r, d+r}$ -perturbation  $(f'_a)_a$  of  $(f_a)_a$  (which is also a  $C_X^{d,r}$ -perturbation). Hence there exists a  $C^{d,\infty}$ -perturbation of  $(f''_a)_a$  of  $(f'_a)_a$  so that  $f''_a$  has a sink of period  $\geq M$  for every  $a \in \mathbb{I}^k$ . By smoothness of the fibration, this is the base dynamics of a  $C_A^{d, r+d}$ -perturbation  $(\hat{f}''_a)_a$  of  $(\hat{f}'_a)_a$ . In particular  $(\hat{f}''_a)_a$  of  $(\hat{f}'_a)_a$  is a  $C_X^{d, r+d}$ -perturbation of  $(\hat{f}'_a)_a$ . Also, since the fibers are contracted, for every  $a \in \mathbb{I}^k$ ,  $\hat{f}''_a$  has a sinks of period  $\geq M$ . This proves the existence of a  $C_X^{d,r}$ -dense set which has a sink of period  $\geq M$  for every  $a \in \mathbb{I}^k$ . Note that this set is necessarily open. The intersection of these open and dense sets among  $M \geq 0$  is the set  $\mathcal{R}$ : a family therein has a sink of arbitrarily large period and so infinitely many sinks, at every parameter  $a \in \mathbb{I}^k$ .  $\square$



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